

Differential Calculus# To Find Expansion By "Maclaurin's Theorem"

Let $y = y(x) = f(x)$ be a function of x which can be extended in ascending power of x , $f(x)$ is differentiable at any no. of time, Then MACLAURIN'S THEOREM is defined as

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

OR

$$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

For example:-

→ Expand $\sin(x)$ by macⁿ The.

Solⁿ:- Let $y = \sin(x)$

$$y(0) = 0$$

$$y_1 = \cos(x)$$

$$y(0) = 1$$

$$y_2 = -\sin(x)$$

$$y_2(0) = 0$$

$$y_3 = -\cos(x)$$

$$y_3(0) = -1$$

$$y_4 = \sin(x)$$

$$y_4(0) = 0$$

$$y_5 = \cos(x)$$

$$y_5(0) = 1$$

Now, A/c Macⁿ The.

$$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$\Rightarrow \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} (-1) + \frac{x^4}{4!} \times 0 + \frac{x^5}{5!} \times 1 + \dots$$

$$\Rightarrow \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Q Expand the given functions by Macⁿ The.

a) $f(x) = a^x$ b) $f(x) = e^x$ c) $f(x) = \cos x$ d) $f(x) = \log(1+x)$

Solⁿ: a) given $\cdot f(x) = a^x \rightarrow f(0) = 1$

$\cdot f'(x) = a^x \ln a$
 $= [\ln a] f(x) \rightarrow f'(0) = \ln a$

$\cdot f''(x) = [\ln a] f'(x) \rightarrow f''(0) = (\ln a)^2$

$\cdot f'''(x) = [\ln a] f''(x) \rightarrow f'''(0) = [\ln a]^3$

Now

By Macⁿ The.

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$[a^x = 1 + \frac{x(\ln a)}{1!} + \frac{x^2(\ln a)^2}{2!} + \frac{x^3(\ln a)^3}{3!} + \dots]$$

Solⁿ: b) given $\cdot f(x) = e^x \rightarrow f(0) = 1$

$\cdot f'(x) = e^x = f(x) \rightarrow f'(0) = 1$

$\cdot f''(x) = f'(x) \rightarrow f''(0) = 1$

$\cdot f'''(x) = f''(x) \rightarrow f'''(0) = 1$

Now A/c Macⁿ The.

$$[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots]$$

Solⁿ: c) given $\cdot f(x) = \cos x \rightarrow f(0) = 1$

$\cdot f'(x) = -\sin x \rightarrow f'(0) = 0$

$\cdot f''(x) = -\cos x \rightarrow f''(0) = -1$

$\cdot f'''(x) = \sin x \rightarrow f'''(0) = 0$

$\cdot f^{(4)}(x) = \cos x = f(x) \rightarrow f^{(4)}(0) = 1$

$\cdot f^{(5)}(x) = -\sin x \rightarrow f^{(5)}(0) = 0$

Now

By Macⁿ The.

$$[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots]$$

Soln: d) $\bullet f(x) = \log(1+x) \rightarrow f(0) = \log(1) = 0$

$\bullet f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$

$\bullet f''(x) = \frac{-1}{(1+x)^2} \rightarrow f''(0) = -1$

$\bullet f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$

$\bullet f^{(4)}(x) = \frac{-6}{(1+x)^4} \rightarrow f^{(4)}(0) = -6$

By Macn The.

$$f(x) = \log(1+x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots$$

$$\Rightarrow [\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots]$$

26/09/24

Q Expand $\log(\sec x)$ by Macn. The. / If $\log(\sec x) = \frac{1}{2}x^2 + Ax^4 + Bx^6$,
Find A, B / Apply Macn The to prove that $\log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots$

Soln: Let $\bullet y = \log(\sec x) \rightarrow y(0) = 0$

$\bullet y_1 = \frac{\sec x \cdot \tan x}{\sec x} = \tan x \rightarrow y_1(0) = 0$

$\bullet y_2 = \sec^2 x \rightarrow y_2(0) = 1$

$\bullet y_3 = 2\sec x \cdot \sec x \tan x = 2y_1 y_2 \rightarrow y_3(0) = 2 \times 0 \times 1 = 0$

$\bullet y_4 = 2[y_1 y_3 + y_2^2] \rightarrow y_4(0) = 2[0 + 1] = 2$

$\bullet y_5 = 2[y_1 y_4 + y_3 y_2 + 2y_2 y_3] \rightarrow y_5(0) = 2[0 + 0 + 0] = 0$

$\bullet y_6 = 2[y_1 y_5 + y_4 y_2 + y_3 y_4 + y_2^2 + 2(y_3 y_3 + y_2 y_4)]$

$= 2[y_1 y_5 + 4y_2 y_4 + 3y_3^2] \rightarrow y_6(0) = 2[0 + 4 \times 2 + 0] = 16$

Now By Macn Theo.

$$y = \log(\sec x) = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \frac{x^6}{6!} y_6(0) + \dots$$

$$\Rightarrow [\log(\sec x) = \frac{1}{2}x^2 + \frac{2x^4}{4!} + \frac{16x^6}{6!} \dots = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots]$$

On Comparing

$$\log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots = \frac{1}{2}x^2 + Ax^4 + Bx^6 + \dots$$

$$A = \frac{1}{12} \mid B = \frac{1}{45}$$

Q Expand $\sin^{-1}x$ by macⁿ The.

Soln: Let $y = \sin^{-1}x \rightarrow \boxed{y(0) = 0}$

$y_1 = \frac{1}{\sqrt{1-x^2}} \rightarrow \boxed{y_1(0) = 1}$

$\Rightarrow (\sqrt{1-x^2})y_1 = 1$

$\Rightarrow y_1 \left[\frac{1}{2}(1-x^2)^{-1/2} \right] + (1-x^2)^{1/2}y_2 = 0$

at $x=0$

$-2x \left(\frac{1}{2}(1-0)^{-3/2} \right) + (1-0)^{1/2}y_2(0) = 0 \rightarrow \boxed{y_2(0) = 0}$

$y_2 = -\frac{1}{2} \frac{(2x)}{(1-x^2)^{3/2}} \rightarrow y_2(0)$

$y_3 = \frac{3x}{2}(1-x^2)^{-5/2} \frac{(-2x)}{x^2} + (1-x^2)^{-3/2} \rightarrow \boxed{y_3(0) = 1}$

$= 3x^2(1-x^2)^{-5/2} + (1-x^2)^{-3/2}$

$= (1-x^2)^{-5/2} [3x^2 + (1-x^2)^{-1}]$

$= (1-x^2)^{-5/2} [3x^2 + 1]$

$y_4 = (1-x^2)^{-5/2} [3+4x] + (3x-x^2+1)(1-x^2)^{-7/2} \frac{(-2x)}{x^2}$

$\rightarrow \boxed{y_4(0) = (1)(3) = 3}$

$y_4 = (1-x^2)^{-7/2} [4x(1-x^2) + (-2x)(3x-x^2+1)]$

$= (1-x^2)^{-7/2} [4x-4x^3+2x^3-6x^2+2x]$

$= (1-x^2)^{-7/2} [0+2x-6x^2-2x^3]$

$y_5 = (1-x^2)^{-7/2} [0+2-12x-6x^2] + (1+4x-6x^2-2x^3)(1-x^2)^{-9/2}$

$\rightarrow y_5(0) = 1 \times 4 = 4$

$$\text{let } y = \sin^{-1} x \rightarrow \boxed{y(0) = 0}$$

$$y_1 = \frac{1}{\sqrt{1-x^2}} \rightarrow \boxed{y_1(0) = 1}$$

$$\Rightarrow (\sqrt{1-x^2}) y_1 = 1$$

$$\Rightarrow (1-x^2) y_1^2 = 1$$

$$\Rightarrow (1-x^2) 2y_1 y_2 + y_1^2 (-2x) = 0$$

$$\Rightarrow y_1 [(1-x^2) y_2 + y_1 (-x)] = 0$$

$$\Rightarrow (1-x^2) y_2 + y_1 (-x) = 0$$

$$\text{at } x=0$$

$$(1-0) y_2(0) + 0(0) = 0$$

$$\Rightarrow \boxed{y_2(0) = 0}$$

again diffⁿ.

$$(1-x^2) y_3 + y_2 (-2x) - (x y_2 + y_1) = 0$$

$$(1-x^2) y_3 - 2x y_2 - x y_2 - y_1 = 0$$

$$(1-x^2) y_3 - 3x y_2 - y_1 = 0$$

$$\text{at } x=0$$

$$(1-0) y_3(0) - 3 \times 0 \times 0 - 1 = 0$$

$$\Rightarrow \boxed{y_3(0) = 1}$$

again diffⁿ.

$$(1-x^2) y_4 + y_3 (-2x) - 3(x y_3 + y_2) - y_2 = 0$$

$$\Rightarrow (1-x^2) y_4 + (-2x y_3) - 3x y_3 - 3y_2 - y_2 = 0$$

$$\Rightarrow (1-x^2) y_4 - 5x y_3 - 4y_2 = 0$$

$$\text{at } x=0$$

$$(1-0) y_4(0) - 0 - 0 = 0$$

$$\Rightarrow \boxed{y_4(0) = 0}$$

again diffⁿ.

$$(1-x^2) y_5 + y_4 (-2x) - 5(x y_4 + y_3) - 4y_3 = 0$$

$$\Rightarrow (1-x^2) y_5 - 2x y_4 - 5x y_4 - 5y_3 - 4y_3 = 0$$

$$\Rightarrow (1-x^2) y_5 - 7x y_4 - 9y_3 = 0$$

$$\text{at } x=0$$

$$(1-0) y_5(0) - 0 - 9 = 0$$

$$\Rightarrow \boxed{y_5(0) = 9}$$

Now

By Macⁿ The.

$$y = \sin^{-1} x = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \dots$$

$$\Rightarrow \left[\sin^{-1} x = x + \frac{x^3}{6} + \frac{9x^5}{120} + \dots \right] //$$

Expand By macⁿ. The.

most imp//

① $y = e^{a \sin^{-1} x}$ and deduce that

$$e^{\theta} = 1 + \sin \theta + \frac{\sin^2 \theta}{2!} + \frac{2}{3!} \sin^3 \theta + \dots$$

Soln:- Given

$$y = e^{a \sin^{-1} x}$$

$$\rightarrow y(0) = 1$$

$$y_1 = e^{a \sin^{-1} x} \times \frac{a}{\sqrt{1-x^2}} = \frac{a e^{a \sin^{-1} x}}{\sqrt{1-x^2}} \rightarrow y_1(0) = a$$

$$\Rightarrow \sqrt{1-x^2} y_1 = a e^{a \sin^{-1} x}$$

$$\Rightarrow (1-x^2) y_1^2 = a^2 y^2$$

diff. w.r.t x

$$2(1-x^2) y_1 y_2 + y_1^2 (-2x) = a^2 2 y y_1$$

$$\Rightarrow (1-x^2) y_2 y_1 = [a^2 + x y_1^2] y_1$$

$$\Rightarrow (1-x^2) y_2 - (a^2 + x y_1^2) = 0$$

$$\left. \begin{array}{l} \text{at } x=0 \\ (1-0) y_2(0) = [a^2 + 0] \end{array} \right\} \Rightarrow y_2(0) = a^2$$

Again, diffⁿ. w.r.t x

$$(1-x^2) y_3 + y_2(-2x) - a^2 y_1 + y_1 = 0 \quad \left| \begin{array}{l} \text{at } x=0 \\ (1-0) y_3(0) + 0 - a^2 \times a - a = 0 \\ \quad \quad \quad +0 \end{array} \right. \Rightarrow y_3(0) = (a^2 + 1)a$$

Now

By macⁿ Theo.

$$e^{a \sin^{-1} x} = 1 + \frac{x}{1!} (a) + \frac{x^2}{2!} (a^2) + \frac{x^3}{3!} (a^2 + 1)a + \dots$$

$$\text{Let } a \sin^{-1} x = \theta$$

$$\sin^{-1} x = \theta/a$$

$$\Rightarrow x = \sin(\theta/a)$$

$$\text{Put } a=1$$

$$\Rightarrow \sin^{-1} x = \theta$$

$$\Rightarrow x = \sin \theta$$

Now

$$e^{\theta} = 1 + \frac{\sin \theta}{1!} (1) + \frac{\sin^2 \theta}{2!} (1) + \frac{\sin^3 \theta}{3!} (1^2 + 1) + \dots$$

$$\Rightarrow [e^{\theta} = 1 + \sin \theta + \frac{\sin^2 \theta}{2!} + \frac{2}{3!} \sin^3 \theta + \dots]$$

(ii) $y = \log(\cos x)$

Soln: given $y = \log(\cos x) \rightarrow y(0) = \log 1 = 0$

$y_1 = \frac{1}{\cos x} \times \sin x = \tan x \rightarrow y_1(0) = 0$

$y_2 = -\sec^2 x \rightarrow y_2(0) = -1$

$y_3 = -2\sec^2 \tan x \rightarrow y_3(0) = 0$
 $y_3 = -2y_2 y_1$

$y_4 = -2[y_1 y_3 + y_2^2] \rightarrow y_4(0) = -2[0 + 1] = -2$

$y_5 = -2[y_1 y_4 + y_3 y_2 + 2y_2 y_3] \rightarrow y_5(0) = -2[0 + 3 \times 0] = 0$

$y_5 = -2[y_1 y_4 + 3y_2 y_3]$

$y_6 = -2[y_1 y_5 + y_4 y_2 + 3(y_3^2 + y_2 y_4)] \rightarrow y_6(0) = -2[0 + 2 + 3(0 + 2)] = -16$

Now

By Mac's Theo.

$y = \log(\cos x) = 0 + \frac{x}{1!} \cdot 0 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-16) + \dots$

$\Rightarrow \log(\cos x) = -\frac{x^2}{2!} + \frac{2x^4}{4!} - \frac{16x^6}{6!} + \dots$

$\Rightarrow \log(\cos x) = -\left[\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots\right]$

(iii) $y = \tan^{-1} x$

Soln: given $y = \tan^{-1} x \rightarrow y(0) = 0$

$y_1 = \frac{1}{1+x^2} \rightarrow y_1(0) = 1$

$y_2 = \frac{-2x}{(1+x^2)^2} \rightarrow y_2(0) = 0$

$y_2 = (-2x)y_1^2$

$y_3 = -2[2xy_1 y_2 + y_1^3] \rightarrow y_3(0) = -2[2 \times 0 + 1] = -2$

$y_3 = -2[y_1(2xy_2 + y_1^3)]$

$y_4 = -2[y_1(2xy_3 + y_2^2) + y_2(2xy_2 + y_1^3)] \rightarrow y_4(0) = -2[1(0+0)+0] = 0$

$$y_4 = -2 \left[2y_1xy_3 + y_1y_2 + y_1y_2 + 2xy_2^2 + y_1y_2 \right] \\ = -2 \left[2[y_1xy_3 + xy_2^2] + 3y_1y_2 + y_2^2 \right] \\ = -2 \left[2x(y_1y_3 + y_2^2) + 3y_1y_2 + y_2^2 \right]$$

$$\bullet y_5 = -2 \left[2x[y_1y_4 + y_3y_2 + y_2^2] + (y_1y_3 + y_2^2)2 + 3(y_1y_3 + y_2^2) \right] \\ \rightarrow y_5(0) = -2 \left[0 + 2[-2+0] + 3[-2+0] + (-2) \right] \\ = -2[-4-6-2] \\ = 24$$

Now

By Macⁿ Theo.

$$y = \tan^{-1} x = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \frac{x^5}{5!} y^{(5)}(0) + \dots$$

$$\Rightarrow \tan^{-1} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\Rightarrow \boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots}$$

Exp^d by Mac. The.

$$\textcircled{1} y = \log(1 + \sin x) \quad \text{ii) } y = \log(1 + \tan x) \quad \text{iii) } y = \frac{e^x}{e^x + 1}$$

$$\text{Soln: } \textcircled{1} y = \log(1 + \sin x) \rightarrow \boxed{y(0) = 0}$$

Taking 'Antilog'

$$e^y = e^{\log(1 + \sin x)}$$

$$\Rightarrow e^y = 1 + \sin x$$

Diffⁿ:

$$e^y \cdot y_1 = 0 + \cos x$$

Again, Diffⁿ:

$$e^y y_2 + y_1 e^y y_1 = -\sin x$$

$$e^y [y_2 + y_1^2] = -\sin x$$

$$\left| \begin{array}{l} \text{at } x=0 \\ e^{y(0)} y_1(0) = \cos 0 \Rightarrow \boxed{y_1 = 1} \end{array} \right.$$

$$\left| \begin{array}{l} \text{at } x=0 \\ e^{y(0)} [y_2(0) + y_1^2(0)] = \sin 0 \\ \Rightarrow \boxed{y_2(0) = -1} \end{array} \right.$$

Again, Diff.

$$e^y [y_3 + 2y_1 y_2] + e^y y_1 (y_2 + y_1^2) = -\cos x$$

at $x=0$
 $y_3(0) + 2 \cdot 1 \cdot (-1) + 1 \cdot (-1+1) = -\cos 0$
 $y_3(0) = 1$

Now

By macⁿ The.

$$y = \log(1 + \sin x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\Rightarrow \log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \dots //$$

Solⁿ (11) Given $y = \log(1 + \tan x) \rightarrow y(0) = 0$

Taking antilog both side

$$e^y = 1 + \tan x$$

Diff. w.r.t x

$$\Rightarrow e^y y_1 = \sec^2 x$$

at $x=0$
 $y_1(0) = y_1 = 1$

Again Diff. w.r.t.

$$e^y [y_2 + y_1^2] = 2 \sec^2 x \tan x$$

$$\Rightarrow e^y [y_2 + y_1^2] = 2(e^y y_1)(e^y - 1) = 2[e^{2y} y_1 - e^y y_1]$$

at $x=0$
 $1[y_2(0) + 1^2] = 2(1 \cdot 1)(1-1)$
 $\Rightarrow y_2(0) = -1$

Again, Diff. w.r.t

$$e^y [y_3 + 2y_1 y_2] + e^y y_1 (y_2 + y_1^2)$$

$$= 2[e^{2y} y_2 + 2y_1 e^{2y} - (e^y y_2 + y_1 e^y)]$$

at $x=0$
 $1[y_3(0) + 2] + 1 \cdot (-1+1) = 2[1+2 - (-1+1)]$
 $y_3(0) = 2$

By macⁿ The.

$$y = \log(1 + \tan x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\Rightarrow \log(1 + \tan x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots //$$

Soln (iii) given $y = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x} \rightarrow \boxed{y(0) = \frac{1}{1+1} = \frac{1}{2}}$

$$y_1 = \frac{e^x}{(1+e^x)} \left(\frac{1}{e^{x+1}} \right)$$

$$\bullet y_1 = y(1-y)$$

$$\rightarrow \boxed{y_1(0) = \frac{1}{2} \left(1 - \frac{1}{2} \right) = \frac{1}{4}}$$

Diffⁿ w.r.t x

$$\bullet y_2 = y_1 - 2yy_1$$

$$\rightarrow \boxed{y_2(0) = \frac{1}{4} - 2 \times \frac{1}{2} \times \frac{1}{4} = 0}$$

$$\bullet y_3 = y_2 - 2(y y_2 + y_1^2)$$

$$\rightarrow \boxed{y_3(0) = 0 - 2 \left(0 + \left(\frac{1}{4} \right)^2 \right) = -\frac{1}{8}}$$

By Mac The.

$$y = \frac{e^x}{1+e^x} = \frac{1}{2} + \frac{1}{4}x - \frac{1}{8 \times 3!}x^3 + \dots$$

$$\Rightarrow \left[\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \dots \right] //$$

20/05/24

Q Expand $(\sin^{-1}x)^2$ By Macⁿ The. and hence prove that

$$\theta^2 = \sin^2\theta + \frac{\sin^4\theta}{3} + \frac{8}{45}\sin^6\theta + \dots$$

Imp

Soln: let $y = (\sin^{-1}x)^2 \rightarrow \boxed{y(0) = 0}$

$$\bullet y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} y_1 = 2\sin^{-1}x \rightarrow \boxed{y_1(0) = 0}$$

Diffⁿ both side

$$\Rightarrow (1-x^2) y_1^2 = 4(\sin^{-1}x)^2$$

$$\Rightarrow (1-x^2) y_1^2 = 4y$$

Diffⁿ w.r.t x

$$(1-x^2) 2y_1 y_2 - 2xy_1^2 = 4y$$

$$\Rightarrow 2y_1[(1-x^2)y_2 - xy_1] = 2(2y_1)$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 2$$

$$\longrightarrow \boxed{y_2(0) = 2}$$

Again, Diffⁿ. w.r.t x

$$\{ (1-x^2)y_3 - 2xy_2 \} - \{ xy_2 + y_1 \} = 0$$

$$\{ (1-x^2)y_3 - 3xy_2 - y_1 \} = 0$$

$$\longrightarrow \boxed{y_3(0) = y_1(0) = 0}$$

Again,

$$\{ (1-x^2)y_4 - 2xy_3 \} - 3\{ xy_3 + y_2 \} = y_2$$

$$(1-x^2)y_4 - 5xy_3 = 4y_2$$

$$\longrightarrow \boxed{y_4(0) = 4y_2(0) = 2 \times 2^2 = 8}$$

Again,

$$\{ (1-x^2)y_5 - 2xy_4 \} - 5\{ xy_4 + y_3 \} = 4y_3$$

$$(1-x^2)y_5 - 7xy_4 = 9y_3$$

$$\longrightarrow \boxed{y_5(0) = 9y_3(0) = 0}$$

Again,

$$\{ (1-x^2)y_6 - 2xy_5 \} - 7\{ xy_5 + y_4 \} = 9y_4$$

$$(1-x^2)y_6 - 9xy_5 = 16y_4$$

$$\longrightarrow \boxed{y_6(0) = 16y_4(0) = 2 \times 2^2 \times 4^2 = 128}$$

Now

By macⁿ. Theo.

$$y = \cancel{y(0)} + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$\Rightarrow [\sin^{-1} x]^2 = 0 + 0 + 2 \cdot \frac{x^2}{2!} + 0 + \frac{x^4}{4!} 8 + 0 + \frac{x^6}{6!} \times 2 \times 2^2 \times 4^2 + \dots$$

$$\Rightarrow (\sin^{-1} x)^2 = \frac{2x^2}{2!} + \frac{8x^4}{4!} + \frac{128x^6}{6!} + \dots$$

Now

$$\text{Put } \sin^{-1} x = \theta \rightarrow (\sin^{-1} x)^2 = \theta^2$$

$$\Rightarrow x = \sin \theta$$

$$\therefore \theta^2 = \frac{2 \sin^2 \theta}{2} + \frac{8 \sin^4 \theta}{24} + \frac{128 \sin^6 \theta}{720} + \dots$$

$$\left[\theta^2 = \sin^2 \theta + \frac{\sin^4 \theta}{3} + \frac{8 \sin^6 \theta}{45} + \dots \right] //$$

Q. Expand $y = e^{\sin x}$ By Macⁿ Theo.

Solⁿ: Given $y = e^{\sin x} \rightarrow \boxed{y(0) = 1}$

$$\bullet y_1 = e^{\sin x} \cdot \cos x \rightarrow \boxed{y_1(0) = 1}$$

$$\Rightarrow y_1 = y \cos x$$

Diff. w.r.t x .

$$\bullet y_2 = -y \sin x + \cos x y_1 \rightarrow \boxed{y_2(0) = -1 \times 0 + 1 \times 1 = 1}$$

$$\bullet y_3 = -[y \cos x + y_1 \sin x] + [\cos x y_2 - y_1 \sin x] \rightarrow \boxed{y_3(0) = 0}$$

Now

By Macⁿ Theo.

$$y = e^{\sin x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^4}{8} + \dots //$$

To find the Expansion By "Taylor Theorem".

Let $f(x+h)$ be a function of x is independent of ' h ', then By "TAYLOR'S THEOREM"

$$\textcircled{1} \boxed{f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots}$$

(Expansion in increasing power of ' h ')

$$\textcircled{2} \boxed{f(h+x) = f(h) + \frac{x}{1!} f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots}$$

(Expansion in increasing power of ' x ')

$$\textcircled{3} \boxed{f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots \quad \boxed{a \in \mathbb{R}}}$$

(Expansion in increasing power of ' $(x-a)$ ')

• NOTE:- Expansion about $x=a$

i.e; Expansion in power of $(x-a)$.

Q Expand $\sin x$ in power of $(x - \pi/2)$

Soln: Expansion in power of $(x - \pi/2)$

$$f(x) = f(\pi/2) + \frac{(x - \pi/2)}{1!} f'(\pi/2) + \frac{(x - \pi/2)^2}{2!} f''(\pi/2) + \dots \quad \text{--- (1)}$$

Let $\bullet f(x) = \sin x \quad \longrightarrow \quad \boxed{f(\pi/2) = \sin(\pi/2) = 1}$

Diffⁿ with x

$\bullet f'(x) = \cos x \quad \longrightarrow \quad \boxed{f'(\pi/2) = 0}$

Again, Diffⁿ with x

$\bullet f''(x) = -\sin x \quad \longrightarrow \quad \boxed{f''(\pi/2) = -1}$

Again,

$\bullet f'''(x) = -\cos x \quad \longrightarrow \quad \boxed{f'''(\pi/2) = 0}$

Again,

$\bullet f^{(4)}(x) = \sin x \quad \longrightarrow \quad \boxed{f^{(4)}(\pi/2) = 1}$

Now, Putting values in eqⁿ (1)

$$f(x) = \sin x = 1 + (x - \pi/2)(0) + \frac{(x - \pi/2)^2}{2}(-1) + \frac{(x - \pi/2)^3}{6}(0) + \frac{(x - \pi/2)^4}{24}(1) + \dots$$

$$\Rightarrow \sin x = 1 + \frac{(x - \pi/2)^2}{2} + \frac{(x - \pi/2)^4}{24} - \frac{(x - \pi/2)^6}{720} + \dots$$

$$\Rightarrow \boxed{\sin x = 1 + \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} - \frac{(x - \pi/2)^6}{6!} + \dots}$$

Q Expand $\log(x)$ in power of $(x-1)$ and hence evaluate $\log(1.1)$.

Soln:- Let $\bullet f(x) = \log(x) \quad \longrightarrow \quad f(1) = 0$

$\bullet f'(x) = \frac{1}{x} \quad \longrightarrow \quad f'(1) = 1$

$\bullet f''(x) = -\frac{1}{x^2} \quad \longrightarrow \quad f''(1) = -1$

$\bullet f'''(x) = \frac{2}{x^3} \quad \longrightarrow \quad f'''(1) = +2$

$\bullet f^{(4)}(x) = -\frac{6}{x^4} \quad \longrightarrow \quad f^{(4)}(1) = -6$

By Taylor Theorem.

$$f(x) = \log(x) = 0 + \frac{(x-1)}{1!}(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \dots$$

$$\Rightarrow \log x = (x-1) + (-1) \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Now

Put $x = 1.1$

$$\log 1.1 = (1.1-1) + (-1) \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \dots$$

$$\begin{aligned} \Rightarrow \log 1.1 &= (0.1) + \frac{(-1)(0.1)^2}{2} + \frac{(0.1)^3}{3} + (-1) \frac{(0.1)^4}{4} + \dots \\ &= 0.1 - 0.005 + 0.0003 - 0.000025 + \dots \end{aligned}$$

$$[\log 1.1 = 0.0953102] \text{ (Approx)}$$

Q. $f(x) = e^x$ in power of $(x-1)$.

Soln:- Given, $f(x) = e^x \rightarrow f(1) = e$

- $f'(x) = e^x \rightarrow f'(1) = e$
- $f''(x) = e^x \rightarrow f''(1) = e$
- $f'''(x) = e^x \rightarrow f'''(1) = e$

By Taylor's Theorem

$$f(x) = e^x = e + \frac{(x-1)}{1!}e + \frac{(x-1)^2}{2!}e + \frac{(x-1)^3}{3!}e + \dots$$

$$\Rightarrow e^x = e \left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \right] //$$

Q. $f(x) = \cos x$ in power of $(x - \pi/2)$.

Soln:- Given, $f(x) = \cos x \rightarrow f(\pi/2) = 0$

- $f'(x) = -\sin x \rightarrow f'(\pi/2) = -1$
- $f''(x) = -\cos x \rightarrow f''(\pi/2) = 0$
- $f'''(x) = +\sin x \rightarrow f'''(\pi/2) = 1$
- $f^{(4)}(x) = \cos x \rightarrow f^{(4)}(\pi/2) = 0$

Now

By Taylor's Theorem:

$$f(x) = \cos x = 0 + \frac{(x-\pi/2)}{1!}(-1) + \frac{(x-\pi/2)^2}{2!}0 + \frac{(x-\pi/2)^3}{3!}(-1) + \frac{(x-\pi/2)^4}{4!}0 + \dots$$

$$\Rightarrow \cos x = -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \frac{(x-\pi/2)^7}{7!} - \dots$$

Q Expand $\tan^{-1}x$ in power of $(x-\pi/4)$

Soln:- Let $y(x) = \tan^{-1}x \rightarrow f(\pi/4) = \tan^{-1}(\pi/4)$

$$y'(x) = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)f'(x) = 1 \rightarrow f'(\pi/4) = \frac{1}{1+(\pi/4)^2} = \frac{16}{16+\pi^2}$$

Diffn. wrt x

$$\Rightarrow (1+x^2)f''(x) + f'(x)(2x) = 0$$

$$\rightarrow f''(\pi/4) = \frac{f'(\pi/4) 2\pi/4}{1+(\pi/4)^2} = \frac{\pi}{2[1+(\pi/4)^2]^2}$$

Now By Taylor's theorem.

$$\tan^{-1}x = \tan^{-1}(\pi/4) + \frac{(x-\pi/4)}{1!} \frac{1}{[1+(\pi/4)^2]} + \frac{(x-\pi/4)^2}{2!} \frac{\pi}{[1+(\pi/4)^2]^2 \times 2} + \dots$$

$$= \frac{168}{16+\pi^2} \times \frac{\pi}{4 \times 16}$$

$$= \frac{128}{(16+\pi^2)^2}$$

$$\Rightarrow \tan^{-1}x = \tan^{-1}(\pi/4) + \frac{(x-\pi/4)}{[1+(\pi/4)^2]} + \frac{(x-\pi/4)^2}{4[1+(\pi/4)^2]^2} + \dots$$

Q Prove that using Taylor's theorem.

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \dots$$

Soln:- By Taylor's Expansion in Power of x .

$$f(x+h) = f(h) + \frac{x}{1!} f'(h) + \frac{x^2}{2!} f''(h) + \dots$$

(1)

Let $f(x+h) = \log(x+h)$

$\Rightarrow f(h) = \log(h)$

Put in eq (1)

$$\log(x+h) = \log(h) + \frac{x}{1!} \frac{1}{h} + \frac{x^2}{2!} \times \frac{(-1)}{h^2} + \frac{x^3}{3!} \frac{2}{h^3} + \dots$$

• $f'(h) = \frac{1}{h}$

• $f''(h) = -\frac{1}{h^2}$

• $f'''(h) = \frac{2}{h^3}$

$$\Rightarrow \log(x+h) = \log(h) + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \dots$$

Proved //

Q Expand $\tan(x+\pi/4)$ as far as the x^4 and hence Evaluate $\tan 46.5^\circ$. Expansion in power of x .

Soln:- Let $f(x+h) = \tan(x+\pi/4) \rightarrow \boxed{h=\pi/4}$

$\Rightarrow f(x) = \tan x \rightarrow f(\pi/4) = \tan \pi/4 = 1$

• $f'(x) = \sec^2 x \rightarrow f'(\pi/4) = 2$

• $f''(x) = 2 \sec^2 x \tan x \rightarrow f''(\pi/4) = 2 \times 2 = 4$

$\Rightarrow f''(x) = 2 f(x) \cdot f'(x)$

• $f'''(x) = 2 f(x) f''(x) + 2 [f'(x)]^2 \rightarrow f'''(\pi/4) = 8 + 8 = 16$

• $f^{(4)}(x) = 2 f(x) f'''(x) + 2 f'(x) f''(x) + 4 f'(x) \cdot f''(x) \rightarrow f^{(4)}(\pi/4) = 32 + 16 + 32 = 80$

Now

By Taylor's theorem

$$f(x+h) = f(h) + \frac{x}{1!} f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{(4)}(h) + \dots$$

$$\tan(x+\pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{5}{3}x^4 + \dots$$

Now

$$\tan 46.5^\circ = \tan(1.5 + 45)$$

$$= \tan\left(\frac{15\pi}{180} + \pi/4\right) = \tan\left(\frac{\pi}{12} + \pi/4\right)$$

$$\Rightarrow \tan(46.5^\circ) = \tan\left(\frac{\pi}{120} + \frac{\pi}{4}\right) = 1 + \frac{2\pi}{120} + \frac{2\left(\frac{\pi}{120}\right)^2}{2!} + \frac{8\left(\frac{\pi}{120}\right)^3}{3!} + \frac{5\left(\frac{\pi}{120}\right)^4}{4!} + \dots$$

$$= 1 + 0.0523 + 0.0013 + 0.000047 + \dots$$

$$\Rightarrow \boxed{\tan(46.5^\circ) = 1.0534}$$

Q Prove that $\tan^{-1}(x+h) = \tan^{-1}(x) + h \sin \theta \frac{\sin \theta}{1!} - \frac{(h \sin \theta)^2 \sin 2\theta}{2!} + \frac{2(h \sin \theta)^3 \sin 3\theta}{3!} + \dots$

Where, $\theta = \cot^{-1} x$

Solⁿ: We know that expansion of $f(x+h)$ in power of h

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \quad \text{--- (1)}$$

Let $f(x+h) = \tan^{-1}(x+h)$

$$\Rightarrow f(x) = \tan^{-1} x$$

$$\begin{aligned} \bullet f'(x) &= \frac{1}{1+x^2} = \frac{1}{1+\cot^2 \theta} = \sin^2 \theta \\ &= (\sin \theta)(\sin \theta) \end{aligned}$$

Given

$$\theta = \cot^{-1} x \rightarrow x = \cot \theta$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{-1}{1+x^2} = \frac{-1}{1+\cot^2 \theta}$$

$$\Rightarrow \left[\frac{d\theta}{dx} = -\sin^2 \theta \right]$$

$$\begin{aligned} \bullet f''(x) &= 2 \sin \theta \cdot \cos \theta \frac{d\theta}{dx} \\ &= -2 \sin 2\theta \sin^2 \theta \end{aligned}$$

$$\begin{aligned} \bullet f'''(x) &= -[\sin^2 \theta \cos 2\theta \times 2 + \sin 2\theta \cdot 2 \cos \theta \sin \theta] \frac{d\theta}{dx} \\ &= -2 \sin \theta [\sin \theta \cos 2\theta + \cos \theta \sin 2\theta] \cdot \sin^2 \theta \\ &= 2 \sin^3 \theta [\sin(2\theta + \theta)] \\ &= 2 \sin^3 \theta \sin 3\theta \end{aligned}$$

Put these values in eq (1)

$$\begin{aligned} f(x+h) &= \tan^{-1}(x+h) = \tan^{-1} x + \frac{(h \sin \theta)(\sin \theta)}{1!} + \frac{h^2 (-\sin^2 \theta)(\sin 2\theta)}{2!} \\ &\quad + \frac{h^3 2 \sin^3 \theta \sin 3\theta}{3!} + \dots \\ \Rightarrow \tan^{-1}(x+h) &= \tan^{-1} x + \frac{(h \sin \theta) \sin \theta}{1!} - \frac{(h \sin \theta)^2 \sin 2\theta}{2!} + \frac{2(h \sin \theta)^3 \sin 3\theta}{3!} + \dots \end{aligned}$$

Q Prove that $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \dots$
By Macⁿ. Theo.

Solⁿ:- Let $y = \log(1+e^x) \rightarrow y(0) = \log 2$

$y_1 = \frac{e^x}{1+e^x} \rightarrow y'(0) = \frac{1}{2}$

$y_2 = \frac{-e^x}{(1+e^x)^2} + \frac{e^x}{(1+e^x)} \rightarrow y_2(0) = \frac{1}{4}$

By, Macⁿ. Theorem.

$$y = y(0) + y'(0)\frac{x}{1!} + y_2(0)\frac{x^2}{2!} + \dots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2} \times \frac{1}{4} + \dots$$

$$\Rightarrow [\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \dots] \text{ Proved}$$

Partial Differentiation [P.D] :-

7/10/24

Let $u = f(x, y)$

Notation "I" :-

$$\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial y} \right] = \frac{\partial}{\partial y} \left[\frac{\partial u}{\partial x} \right]$$

"II" i) $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \rightarrow u$ diffⁿ p.w.r.t. x , keeping y as const.

ii) $\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \rightarrow u$ diffⁿ p.w.r.t. y , keeping x as const.

Similarly,

Let $u = f(x, y, z)$

$$\Rightarrow \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} \right) \right] = \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial z} \right) \right]$$

i) $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \rightarrow u$ diffⁿ p.w.r.t. x , keeping $y, z \rightarrow$ const.

ii) $\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \rightarrow u$ diffⁿ p.w.r.t. y keeping $x, z \rightarrow$ const.

iii) $\frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2} \rightarrow u$ diffⁿ p.w.r.t. z keeping x, y as const.

Ex:- If $u = x^3 + y^3 - 3axy$ then verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Soln:- Given $u = x^3 + y^3 - 3axy$

i) Diffⁿ p.w.r.t. x keeping y as const.

$$\frac{\partial u}{\partial x} = 3x^2 + 0 - 3ay = 3x^2 - 3ay \quad \text{--- (i)}$$

ii) Diffⁿ p.w.r.t. y keeping x as const.

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax \quad \text{--- (ii)}$$

Now

$$LHS = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial y} \right] = \frac{\partial}{\partial x} (3y^2 - 3ax) = -3a \quad \text{--- (iii)}$$

and,

$$RHS = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial y} (3x^2 - 3xy) = -3x \quad \text{--- (iv)}$$

from (iii) and (iv)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad // \text{verified}$$

Ex:- If $u = f(y/x)$. Then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$,

Soln: Given $u = f(y/x)$

Diff. p.w.r.t x

$$\frac{\partial u}{\partial x} = [f'(y/x)] [y(-1/x^2)]$$

$$\Rightarrow x \frac{\partial u}{\partial x} = -y/x [f'(y/x)] \quad \text{--- (i)}$$

Again, Diff. p.w.r.t y

$$\frac{\partial u}{\partial y} = [f'(y/x)] [1/(x)]$$

$$\Rightarrow y \frac{\partial u}{\partial y} = \frac{y}{x} [f'(y/x)] \quad \text{--- (ii)}$$

On adding (i) and (ii)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -y/x [f'(y/x)] + y/x [f'(y/x)] = 0$$

proved

Q If $u = e^{xyz}$ then prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2 + z^2)e^{xyz}$.

Soln: Given $e^{xyz} = u$

Diff. p.w.r.t z

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot yx = x [e^{xyz} \cdot y]$$

Now, Diff. p. w.r.t y

$$\frac{\partial^2 u}{\partial y \partial z} = x[e^{xyz} + y e^{xyz} \cdot xz] \\ = e^{xyz}[x + x^2 yz]$$

Again, Diff. p. w.r.t x

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}[1 + 2xyz] + (x + x^2 yz) e^{xyz} \cdot yz \\ = e^{xyz}[1 + 2xyz + xyz + x^2 y^2 z^2]$$

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}[1 + 3xyz + x^2 y^2 z^2] \text{ . Proved} //$$

Q. If $u = (x^2 + y^2 + z^2)^{-1/2}$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Soln: Given $u = (x^2 + y^2 + z^2)^{-1/2}$

Diff. p. w.r.t. x

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \times 2x = -x (x^2 + y^2 + z^2)^{-3/2}$$

Again, Diff. p. w.r.t x.

$$\frac{\partial^2 u}{\partial x^2} = -\left[x \cdot -\frac{3}{2} (x^2 + y^2 + z^2)^{-5/2} \times 2x + (x^2 + y^2 + z^2)^{-3/2} \cdot (1) \right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = [3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \dots (i)$$

Similarly;

$$\frac{\partial^2 u}{\partial y^2} = [3y^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \dots (ii)$$

and,

$$\frac{\partial^2 u}{\partial z^2} = [3z^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \dots (iii)$$

On adding (i), (ii) and (iii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = [3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-5/2} - 3(x^2 + y^2 + z^2)^{-3/2}]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = [3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}]$$

$$= 0 \quad \text{proved} //$$

Q. If $u = \begin{vmatrix} x^2+y^2+z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Soln: Given $u = x^2(y-z) - y^2(x-z) + z^2(x-y)$

$$\Rightarrow u = x^2y - x^2z - y^2x + y^2z + z^2x - z^2y$$

$$\bullet \frac{\partial u}{\partial x} = 2xy - 2xz - y^2 + z^2 \quad \text{--- (i)}$$

$$\bullet \frac{\partial u}{\partial y} = x^2 - 2yx + 2yz + (-z^2) \quad \text{--- (ii)}$$

$$\bullet \frac{\partial u}{\partial z} = -x^2 + y^2 + 2zx - 2zy \quad \text{--- (iii)}$$

On adding (i), (ii) and (iii)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{2xy} - \cancel{2xz} - \cancel{y^2} + \cancel{z^2} + \cancel{x^2} - \cancel{2yx} + \cancel{2yz} - \cancel{z^2} + \cancel{y^2} - \cancel{x^2} + \cancel{2zx} - \cancel{2zy} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \quad \text{proved} //$$

Q. Given $u = \log(x^3+y^3+z^3-3xyz)$, prove that

$$i) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$ii) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = -\frac{9}{(x+y+z)^2}$$

Soln: 1) $u = \log(x^3+y^3+z^3-3xyz)$

$$\bullet \frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} \quad \text{--- (1)}$$

$$\bullet \frac{\partial u}{\partial y} = \frac{3(y^2 - xz)}{x^3 + y^3 + z^3 - 3xyz} \quad \text{--- (ii)}$$

$$\bullet \frac{\partial u}{\partial z} = \frac{3(z^2 - yx)}{x^3 + y^3 + z^3 - 3xyz} \quad \text{--- (iii)}$$

On adding (i), (ii) and (iii).

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \text{ proved} //$$

Soln: (ii) $\frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}$, $\frac{\partial u}{\partial y} = \frac{3(y^2 - xz)}{x^3 + y^3 + z^3 - 3xyz}$, $\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$

Now

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{3}{x+y+z} \right) \quad \text{--- from (i)}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = 3 \left[\frac{\partial (x+y+z)^{-1}}{\partial x} + \frac{\partial (x+y+z)^{-1}}{\partial y} + \frac{\partial (x+y+z)^{-1}}{\partial z} \right]$$

$$= 3 \left[-(x+y+z)^{-2} + (-1)(x+y+z)^{-2} - (x+y+z)^{-2} \right]$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = -9 (x+y+z)^{-2} \text{ proved} //$$

Q If $u = (x^2 + y^2 + z^2)^{-1/2}$ then prove that

$$\textcircled{i} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -u$$

$$\textcircled{ii} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Soln: Given $u = (x^2 + y^2 + z^2)^{-1/2}$

$$(i) \quad \frac{\partial u}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\Rightarrow x \frac{\partial u}{\partial x} = -x^2(x^2 + y^2 + z^2)^{-3/2} \quad \text{--- (i)}$$

Similarly:

$$y \frac{\partial u}{\partial y} = -y^2(x^2 + y^2 + z^2)^{-3/2} \quad \text{--- (ii)}$$

$$\text{and, } z \frac{\partial u}{\partial z} = -z^2(x^2 + y^2 + z^2)^{-3/2} \quad \text{--- (iii)}$$

On adding (i), (ii) and (iii)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-3/2}$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -(x^2 + y^2 + z^2)^{-1/2} = -u$$

Proved //

$$(ii) \quad \frac{\partial^2 u}{\partial x^2} = -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\left[-\frac{3x}{2}(x^2 + y^2 + z^2)^{-5/2}(2x) + (x^2 + y^2 + z^2)^{-3/2}\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = [3x^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \text{--- (iv)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = [3y^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \text{--- (v)}$$

$$\text{and, } \frac{\partial^2 u}{\partial z^2} = [3z^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}] \quad \text{--- (vi)}$$

On adding (iv), (v) and (vi)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = [3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-5/2} - 3(x^2 + y^2 + z^2)^{-3/2}]$$

$$= [3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}] = 0$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \quad \text{Hence proved}$$

most Imp

Q. If $x^x y^y z^z = c$ Then show that, at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log e x)^{-1}$$

Soln: [Accn to Problem] $z = f(x, y)$ Given $x^x y^y z^z = c$
Taking "log" on both sides
 $\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right]$ $\log(x^x y^y z^z) = \log c$
 $\Rightarrow x \log x + y \log y + z \log z = \log c$

diff'n p.wrt x we have

$$(x \times \frac{1}{x} + \log x) + 0 + (z \times \frac{1}{z} + \log z) \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = - \left(\frac{1 + \log x}{1 + \log z} \right)$$

Similarly

$$\frac{\partial z}{\partial y} = - \left(\frac{1 + \log y}{1 + \log z} \right)$$

Now

$$\begin{aligned} \text{LHS} = \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right] = \frac{\partial}{\partial x} \left[- \left(\frac{1 + \log y}{1 + \log z} \right) \right] \\ &= + (1 + \log y) \frac{1}{z} (1 + \log z)^{-2} \frac{\partial z}{\partial x} \end{aligned}$$

$$= + \frac{1}{z} \left(\frac{1 + \log y}{1 + \log z} \right) x - \left(\frac{1 + \log x}{1 + \log z} \right)$$

Now

at $x=y=z$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= - \frac{1}{x} \left(\frac{1 + \log x}{1 + \log x} \right) x \frac{1 + \log x}{1 + \log x} = - [x (1 + \log x)]^{-1} \\ &= - x^{-1} (\log e + \log x)^{-1} \end{aligned}$$

$$\Rightarrow \text{LHS} = - [x \log e x]^{-1} = \text{RHS} \quad \text{Proved}$$

Q. most imp If $u = f(r)$, where $r^2 = x^2 + y^2$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

Soln: Given $u = f(r)$:

Diffⁿ p.w.r.t x

$$\frac{\partial u}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

Given

$$r^2 = x^2 + y^2$$

Diffⁿ p.w.r.t x

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$\frac{\partial u}{\partial x} = f'(r) \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = x \left[\frac{f'(r)}{r} \right]$$

Again, Diff. p.w.r.t x .

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + x \left[\frac{r f''(r) - f'(r)}{r^2} \right] \frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + \frac{x}{r^2} [r f''(r) - f'(r)] \frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{x^2}{r^2} \left[f''(r) - \frac{1}{r} f'(r) \right] \quad \text{--- (I)}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r} + \frac{y^2}{r^2} \left[f''(r) - \frac{1}{r} f'(r) \right] \quad \text{--- (II)}$$

On Adding (I) and (II).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 \frac{f'(r)}{r} + \frac{x^2 + y^2}{r^2} \left[f''(r) - \frac{1}{r} f'(r) \right]$$

$$= 2 \frac{f'(r)}{r} + \frac{r^2}{r^2} \left[f''(r) - \frac{1}{r} f'(r) \right]$$

$$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r) \right]$$

Proved

Q If $z(x+y) = x^2 + y^2$ show that:-

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Soln:- $z = f(x, y)$

Given $z(x+y) = x^2 + y^2$

$$\Rightarrow z = \frac{x^2 + y^2}{x+y}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2 + y^2)}{(x+y)^2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} = \frac{x^2 - y^2 + 2xy}{(x+y)^2}$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

Now

on Putting LHS = RHS

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right]^2 = 4\left[1 - \left(\frac{x^2 + 2xy - y^2}{(x+y)^2} + \frac{y^2 + 2xy - x^2}{(x+y)^2}\right)\right]$$

$$\Rightarrow \left[\frac{2(x^2 + y^2)}{(x+y)^2}\right]^2 = 4\left[1 - \frac{4xy}{(x+y)^2}\right]$$

$$\Rightarrow \left[\frac{2(x+y)(x-y)}{(x+y)(x+y)}\right]^2 = 4\left[\frac{(x+y)^2 - 4xy}{(x+y)^2}\right]$$

$$\Rightarrow \frac{4(x-y)^2}{(x+y)^2} = 4\frac{(x-y)^2}{(x+y)^2}$$

Proved //

Q If $u = \log (\tan x + \tan y + \tan z)$, then show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Soln:- Given $u = \log (\tan x + \tan y + \tan z)$

Diff. P. w.r.t x

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\Rightarrow \sin 2x \frac{\partial u}{\partial x} = \frac{2 \sin x \cos x \sec x \sec x}{\tan x + \tan y + \tan z} = \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad \text{--- (i)}$$

Similarly

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad \text{--- (ii)}$$

$$\text{and } \sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad \text{--- (iii)}$$

On (i) + (ii) + (iii)

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2$$

Imp Q $\theta = x^n e^{-\frac{x^2}{4t}}$. What is value of n if $\frac{1}{x^2} \left[\frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta}{\partial x} \right) \right] = -\frac{\partial \theta}{\partial t}$. proved //

$$\text{Soln: Given } \frac{1}{x^2} \left[\frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta}{\partial x} \right) \right] = -\frac{\partial \theta}{\partial t} \quad \text{--- (i)}$$

Also Given

$$\theta = x^n e^{-\frac{x^2}{4t}}$$

Diff. P. w.r.t x

$$\frac{\partial \theta}{\partial x} = x^n e^{-\frac{x^2}{4t}} \left(-\frac{x}{2t} \right) - \frac{1}{x^2} + e^{-\frac{x^2}{4t}} (n x^{n-1})$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = x^n e^{-\frac{x^2}{4t}} \left[\frac{x^2}{4t} + \frac{n}{x} \right]$$

$$\Rightarrow \frac{\partial \theta}{\partial r} = \theta \left[\frac{r^2}{4t} + \frac{\eta}{t} \right] \quad \text{--- (ii)}$$

Again, diff. θ p. wrt r

$$\Rightarrow \frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \left(-\frac{1}{4t} \right) 2r$$

$$\Rightarrow \frac{\partial \theta}{\partial r} = -\frac{\theta r}{2t} \quad \text{--- (iii)}$$

multiply by r^2

$$\Rightarrow r^2 \frac{\partial \theta}{\partial r} = -\frac{\theta r^3}{2t}$$

diff. p. wrt r

$$\begin{aligned} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) &= -\frac{1}{2t} [3r^2 \theta + r^3 \frac{\partial \theta}{\partial r}] \\ &= -\frac{1}{2t} [3r^2 \theta + r^3 \left(-\frac{\theta r}{2t} \right)] \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) = r^2 \theta \left[-\frac{3}{2t} + \frac{r^2}{4t} \right]$$

dividing by r^2

$$\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) \right] = \theta \left[-\frac{3}{2t} + \frac{r^2}{4t} \right] \quad \text{--- (iv)}$$

on equating (ii) and (iv)

$$\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) \right] = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \theta \left[-\frac{3}{2t} + \frac{r^2}{4t} \right] = \theta \left[\frac{\eta}{t} + \frac{r^2}{4t} \right]$$

On Comparing LHS = RHS

$$\boxed{\eta = -\frac{3}{2}}$$

Homogenous Function:-

A function $f(x, y) = x^n f(y/x)$ is said to be Homogenous function of degree "n"

Ex:- $u = \left(\frac{x^3 + y^3}{x + y} \right)$ $\begin{matrix} \text{N degree} = 3 \\ \text{D degree} = 1 \end{matrix}$

$$\Rightarrow u = \frac{x^3}{x} \frac{(1 + y^3/x^3)}{(1 + y/x)}$$

$$\Rightarrow u = x^2 f(y/x)$$

Hence, $\boxed{n=2}$

Ex:- $u = \left[\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}} \right]^{1/2}$

$$\Rightarrow u = x^{1/2} f(y/x)$$

$$\Rightarrow \boxed{n = 1/2}$$

→ Direct method-

$$\left[\begin{array}{l} n = \text{degree of numerator} \\ - \text{degree of denominator} \end{array} \right]$$

Ex ① $n = 3 - 1 = 2$

② $n = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

Euler's Formula/Theorem:-

[Inverse Trigonometry & Log Problem]

• If $u = f(x, y) = x^n f(y/x)$ be a homogenous function then

Euler's Theorem is defined as-

$$\boxed{x \frac{\partial}{\partial x} f(x, y) + y \frac{\partial}{\partial y} f(x, y) = n f(x, y)}$$

$$\Rightarrow x f'(u) \frac{\partial u}{\partial x} + y f'(u) \frac{\partial u}{\partial y} = n f(u)$$

$$\Rightarrow f'(u) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = n f(u)$$

→ Deduction Formula:-

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}}$$

→ Euler's Theorem For 2nd order Partial Differentiation.

→ Deduction Formula:-

where

$$g(u) = \frac{n f(u)}{f'(u)}$$

$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u) [g'(u) - 1]}$$

Ex:- If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$. Then show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$.

Soln:- Given

$$u = \tan^{-1} \frac{x^3+y^3}{x-y}$$

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} \quad \boxed{n = 3-1 = 2}$$

We have Euler's Reduction formula

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u$$

Proved //

also Show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u$

By Euler's theorem

Soln: $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u) [g'(u) - 1]$

$$= \sin 2u [2 \cos 2u - 1]$$

$$= 2 \sin 2u \cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$\left[g(u) = \frac{nf(u)}{f'(u)} \right]$$

$$= \sin 2u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u$$

$$\left[\begin{aligned} \sin C - \sin D \\ = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \end{aligned} \right]$$

Proved //

Q If $u = \sin^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$ Then prove that

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin 4u \cos 2u}{4 \cos 3u}$

Soln:- Given $u = \sin^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$

$$\Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\left\{ n = 1 - \frac{1}{2} = \frac{1}{2} \right\}$$

We have by Euler's theorem

$$\textcircled{i} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = g(u)$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u} = \frac{1}{2} \tan u$$

Proved

\textcircled{ii} Again by Euler's reduction formula.

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= g(u) \{g'(u) - 1\} \\ &= \frac{1}{2} \tan u \left[\frac{1}{2} \sec^2 u - 1 \right] \\ &= \frac{1}{2} \left[\frac{1}{2} \frac{\sin u}{\cos^3 u} - \frac{\sin u}{\cos u} \right] \\ &= \frac{1}{2} \left[\frac{\sin u - 2 \sin u \cos^2 u}{2 \cos^3 u} \right] \\ &= \frac{1}{2} \left[\frac{\sin u [1 - 2 \cos^2 u]}{2 \cos^3 u} \right] \\ &= \frac{1}{2} \left[\frac{\sin u \cos 2u}{2 \cos^3 u} \right] \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{1}{4} \left[\frac{\sin u \cos 2u}{\cos^3 u} \right]$$

Proved

Q. If $u = \log \frac{x^4 + y^4}{x+y}$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$$

Soln: Given $u = \log \frac{x^4 + y^4}{x+y}$

$$\Rightarrow \text{antilog } u = \frac{x^4 + y^4}{x+y}$$

$$\boxed{n = 4 - 1 = 3}$$

$$\Rightarrow e^u = \frac{x^4 + y^4}{x+y}$$

By Euler's reduction

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = 3 \frac{e^u}{e^u} = 3$$

Proved

Q If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u$$

Soln:- Given $u = \sec^{-1} \frac{x^3-y^3}{x+y}$

$$\Rightarrow \sec u = \frac{x^3-y^3}{x+y}$$

$$\boxed{n = 3-1 = 2}$$

By Euler's reduction

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \sec u}{\tan u \sec u} = \frac{2}{\tan u} = 2 \cot u \quad \text{Proved} //$$

Q If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$ then prove that

$$(i) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \sin 2u$$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin 2u$$

Soln:- Given $u = \tan^{-1}\left(\frac{y^2}{x}\right)$

$$\Rightarrow \tan u = \frac{y^2}{x} = x \left(\frac{y}{x}\right)^2 = x \left(\frac{y}{x}\right) \quad \boxed{n = 1}$$

Now, By Euler's reduction

$$(i) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1 \frac{\tan u}{\sec^2 u} = \frac{\sin u}{\cos u} \times \cos^2 u = \frac{2 \sin u \cos u}{2} = \frac{1}{2} \sin 2u$$

Proved //

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \sin 2u \left[\frac{1}{2} \times 2 \cos 2u - 1 \right]$$

$$= \frac{1}{2} [\sin 2u \cos 2u - \sin 2u]$$

$$= \frac{1}{2} \sin 2u (\cos 2u - 1)$$

$$= -\frac{1}{2} \sin 2u \times 2 \sin^2 u$$

$$= -\sin 2u \sin^2 u$$

$$g(u) = \frac{1}{2} \sin 2u$$

Proved //

Q If $\sin u = \frac{x^2 y^2}{x+y}$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3 \tan u$$

Soln: given $\sin u = \frac{x^4 (y/n)^2}{x(1+y/n)} = \frac{x^3 (y/n)^2}{1+y/n}$ $n=3$

By Euler's reduction

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3 \frac{\sin u}{\cos u} = 3 \tan u$$

Proved

Q If $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ Verify Euler's Theorem OR Prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{20} u$$

Soln: Given

$$u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$\boxed{n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}}$$

We have to show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{20} u$$

Now,

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5} + y^{1/5}) \left(\frac{1}{4} x^{1/4-1} \right) - (x^{1/4} + y^{1/4}) \left(\frac{1}{5} x^{1/5-1} \right)}{(x^{1/5} + y^{1/5})^2}$$

$$\Rightarrow x \frac{\partial u}{\partial x} = \frac{\left(\frac{1}{4} x^{1/4} \right) (x^{1/5} + y^{1/5}) - \left(\frac{1}{5} x^{1/5} \right) (x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2} \quad \text{--- (i)}$$

Similarly

$$y \frac{\partial u}{\partial y} = \frac{\left(\frac{1}{4} y^{1/4} \right) (x^{1/5} + y^{1/5}) - \left(\frac{1}{5} y^{1/5} \right) (x^{1/4} + y^{1/4})}{(x^{1/5} + y^{1/5})^2} \quad \text{--- (ii)}$$

On (i) + (ii)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\frac{1}{4}(x^4+y^4)(x^5+y^5) - \frac{1}{5}(x^5+y^5)(x^4+y^4)}{(x^5+y^5)^2}$$

$$= \frac{(x^4+y^4)(x^5+y^5) \left[\frac{1}{4} - \frac{1}{5} \right]}{(x^5+y^5)(x^5+y^5)}$$

$$= \frac{1}{20} \left[\frac{x^4+y^4}{x^5+y^5} \right]$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{20} u //$$

Proved

Maxima and Minima (For two Variable)

Let $u = f(x, y)$

To discuss the maxima and minima for u

• Working Rule:-

(i) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

(ii) For maxima and minima

$\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0 \rightarrow$ We get two Equations between x and y
On solving for x and y , we get points
say $(a_1, b_1), (a_2, b_2), \dots$

These points are called Stationary Points

(iii) Find $r = \frac{\partial^2 u}{\partial x^2}$; $s = \frac{\partial^2 u}{\partial x \partial y}$; $t = \frac{\partial^2 u}{\partial y^2}$ at given points respectively.

Now

- If $rt - s^2 > 0$; $r > 0 \rightarrow$ "u" has minima at given point
- If $rt - s^2 > 0$; $r < 0 \rightarrow$ "u" has maxima at given point
- If $rt - s^2 < 0$; $\rightarrow$ "u" has neither maxima nor minima.
- If $rt - s^2 = 0$; $\rightarrow$ Case Undecided, further investigation is needed.

Ex:- P₁ Discuss the maxima and minima for $u = x^3 + y^3 + t - 3axy$

Soln, I) $\frac{\partial u}{\partial x} = 3x^2 - 3ay$; $\frac{\partial u}{\partial y} = 3y^2 - 3ax$

II) $\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0$; $\frac{\partial u}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0$
 $\Rightarrow \boxed{x^2 = ay}$ - (i) $\Rightarrow \boxed{y^2 = ax}$ - (ii)

Sq. (i) $x^4 = a^2 y^2 \Rightarrow y^2 = \frac{x^4}{a}$
Put in (ii)

$\Rightarrow \frac{x^4}{a} = ax \Rightarrow x(x^3 - a^2) = 0$
 $\Rightarrow x = 0$; $x = a$

from (ii)

at $x = 0, y = 0$
 $x = a, y = a$

Points are $(0, 0)$ & (a, a)

III) $r = \frac{\partial^2 u}{\partial x^2} = 6x$, $s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} [3y^2 - 3ax] = -3a$, $t = 6y$

Now

• at (a, a)

$$r = 6a, s = -3a, t = 6a$$

$$\therefore rt - s^2 = 36a^2 - (-3a)^2 = 27a^2$$

$$\Rightarrow \boxed{rt - s^2 > 0}$$

and $\boxed{r > 0}$ hence, u has minima at (a, a)

a, s, t
Always $\oplus$
 $a > 0$
 $a < 0$

• at $(0, 0)$

$$r = 0, s = -3a, t = 0$$

$$\therefore rt - s^2 = -9a^2 < 0$$

hence, u has neither maxima nor minima at $(0, 0)$

Q. Discuss maxima and minima for $u = x^3 - 4xy + 2y^2$

Soln: $u = x^3 - 4xy + 2y^2$

$$\bullet \frac{\partial u}{\partial x} = 3x^2 - 4y$$

$$\bullet \frac{\partial u}{\partial y} = 4y - 4x$$

For find the stationary points

$$\frac{\partial u}{\partial x} = 0 \Rightarrow 3x^2 - 4y = 0 \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow 4y - 4x = 0 \Rightarrow x = y \quad \text{--- (ii)}$$

Put value of x in eq (i)

$$\Rightarrow 3y^2 - 4y = 0$$

$$\Rightarrow y(3y - 4) = 0$$

$$\Rightarrow y = 0 \mid y = \frac{4}{3}$$

$$\therefore x = y$$

$$\therefore x = 0 \mid x = \frac{4}{3}$$

Therefore points are, $(0, 0), (\frac{4}{3}, \frac{4}{3})$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} [4y - 4x] = -4$$

$$t = \frac{\partial^2 u}{\partial y^2} = 4$$

• At $(\frac{4}{3}, \frac{4}{3})$

$$r = 8 > 0 \mid rt - s^2 = 32 - 16 > 0$$

∴ u has minima at $(x, y) = (\frac{4}{3}, \frac{4}{3})$

$$\begin{aligned} u_{\min} &= \left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) + 2\left(\frac{4}{3}\right)^2 \\ &= \left(\frac{4}{3}\right)^2 \frac{4}{3} - 2\left(\frac{4}{3}\right)^2 \\ &= \frac{1}{3}\left(\frac{4}{3}\right)^2 = \frac{16}{27} \end{aligned}$$

* At $(0, 0)$

$$x^2 - y^2 = 0 - 4 < 0$$

u has neither maxima nor minima.

Q Discuss maxima and minima for given function, also find their maximum and minimum value if possible

i) $u = \sin x + \sin y + \sin(x+y)$ iv) $u = \sin x \sin y \sin(x+y)$

ii) $u = xy + x^3 + y^3$ v) $u = x^3 y^2 (1-x-y)$

iii) $u = xy(a-x-y)$ vi) $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Soln: i) $u = \sin x + \sin y + \sin(x+y)$

$$\frac{\partial u}{\partial x} = 0, \cos x + \cos(x+y) = 0 \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} = 0, \cos y + \cos(x+y) = 0 \quad \text{--- (ii)}$$

on comparing (i) and (ii)

$$\cos x = \cos y \Rightarrow x = y$$

Put in eq (i)

$$\cos x + \cos 2x = 0$$

$$\Rightarrow 2\cos x + 2\cos^2 x - \cos x - 1 = 0$$

$$\Rightarrow 2\cos x (\cos x + 1) - 1(\cos x + 1) = 0$$

$$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$$

$$\Rightarrow \cos x = \frac{1}{2} \quad | \quad \cos x = -1 \quad \text{∴ } x = \frac{\pi}{3}$$

$$\Rightarrow x = \frac{\pi}{3} \quad | \quad x = \pi \quad \text{∴ } y = \frac{\pi}{3} \quad | \quad y = \pi$$

We have two stationary points are $(\pi/3, \pi/3), (\pi, \pi)$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\sin(x+y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \sin(x+y)$$

○ At point $(\pi/3, \pi/3)$

$$r = -\sqrt{3}$$

$$rt - s^2 = 3 - \frac{3}{4} > 0 \quad | \quad r < 0$$

$$s = -\sqrt{3}/2$$

$$t = -\sqrt{3}$$

∴ u has maxima at $(\pi/3, \pi/3)$

$$\begin{aligned} \Rightarrow \underline{u_{\max}} &= \sin \pi/3 + \sin \pi/3 + \sin(2\pi/3) \\ &= 2\sqrt{3}/2 + \sqrt{3}/2 \\ &= 3\sqrt{3}/2 \end{aligned}$$

○ At point (π, π)

$$r = 0$$

$$rt - s^2 = 0$$

$$s = 0$$

∴ Case unidentified, further investigation

$$t = 0$$

is needed for ' u ' at (π, π)

Soln. (II) $u = xy + \frac{y^3}{x} + \frac{x^3}{y}$

$$\bullet \frac{\partial u}{\partial x} = 0, \quad y - \frac{y^3}{x^2} = 0$$

$$\Rightarrow y = +\frac{y^3}{x^2} \quad \text{--- (1)}$$

Put in (2)

$$\bullet \frac{\partial u}{\partial y} = 0, \quad x - \frac{x^3}{y^2} = 0$$

--- (2)

$$x - \frac{x^3}{(\frac{y^3}{x^2})^2} = 0 \Rightarrow x - \frac{x^4}{y^3} = 0 \Rightarrow x(y^3 - x^3) = 0 \Rightarrow x = 0, +y$$

When

$$x = +a$$

$$y = +\frac{a^3}{(+a)^2} = +a$$

We have stationary point

$$(+a, +a)$$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3} \quad | \quad s = \frac{\partial^2 u}{\partial x \partial y} = 1 \quad | \quad t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}$$

⊙ At point $(+a, +a)$

$$rt - s^2 = (+2)(+2) - 1 > 0 \quad | \quad r = +2 > 0$$

∴ u has minima at $(+a, +a)$

$$\Rightarrow [u]_{\min} = (+a)^2 + \frac{a^3}{(+a)} + \frac{a^3}{(+a)} = 3a^2 //$$

⊙ At point $(0, 0)$

$$rt - s^2 = \text{undefined.}$$

Soln III

$$u = xy(a - x - y) = axy - x^2y - xy^2$$

$$\bullet \frac{\partial u}{\partial x} = 0, \quad ay - 2xy - y^2 = 0 \quad - \textcircled{i}$$

$$\bullet \frac{\partial u}{\partial y} = 0, \quad ax - x^2 - 2xy = 0 \quad - \textcircled{ii}$$

On $\textcircled{ii} - \textcircled{i}$

$$a(x - y) + (y^2 - x^2) = 0$$

$$\Rightarrow (x - y)[a + (x + y)] = 0$$

$$\Rightarrow x = y = 0$$

$$\Rightarrow x = y \quad - \textcircled{iv}$$

Put in eq $\textcircled{i}$

$$\Rightarrow ay - 2y^2 - y^2 = 0$$

$$\Rightarrow y(a - 3y) = 0$$

$$y = 0 \quad | \quad y = \frac{a}{3} \quad \text{also } x = 0, x = \frac{a}{3}$$

$$a - (x + y) = 0$$

$$x = a - y \quad - \textcircled{v}$$

Put in eq $\textcircled{i}$

$$\Rightarrow ay - 2(a - y)y - y^2 = 0$$

$$\Rightarrow ay - 2ay + 2y^2 - y^2 = 0$$

$$\Rightarrow y(y - a) = 0 \Rightarrow y = 0, a$$

$$x = a, 0$$

We have stationary points are
 $(0,0), (0,a), (a,0), (a/3, a/3)$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = -2y$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y$$

$$t = -2x$$

① At $(a/3, a/3)$

$$r = -2a/3$$

$$s = a - 2a/3 - 2a/3 = -a/3$$

$$t = -2a/3$$

$$\begin{aligned} \bullet \quad r t - s^2 &= \frac{4}{9} a^2 - \frac{a^2}{9} > 0 \quad \left| \begin{array}{l} \text{Suppose} \\ a = +ve \end{array} \right. \\ \bullet \quad r &= -2a/3 < 0 \end{aligned}$$

$\therefore u$ has maxima at $(a/3, a/3)$

$$\Rightarrow \underline{u_{\max}} = \frac{a}{3} \cdot \frac{a}{3} (a - a/3 - a/3) = \left(\frac{a}{3}\right)^3$$

② At $(a, 0)$

$$r = +0$$

$$s = a - 2a = -a$$

$$t = -2a$$

$$\bullet \quad r t - s^2 = 0 - a^2 < 0$$

$\therefore u$ has neither maxima nor minima

Similarly

③ At $(0, a)$

'u' has neither maxima nor minima.

and

④ At $(0, 0)$

$$r = 0$$

$$s = a$$

$$t = 0$$

$$\bullet \quad r t - s^2 = 0 - a^2$$

$\therefore u$ has neither maxima nor minima.

Soln (iv) $u = \sin x \sin y \sin(x+y)$

$$\begin{aligned} \bullet \frac{\partial u}{\partial x} &= 0, \quad \sin y [\sin x \cos(x+y) + \sin(x+y) \cos x] = 0 \\ &\Rightarrow \sin y \sin(2x+y) = 0 \\ &\Rightarrow y=0, \quad 2x+y = \pi \quad \text{--- (i)} \end{aligned}$$

$$\begin{aligned} \bullet \frac{\partial u}{\partial y} &= 0, \quad \sin x [\sin(x+2y)] = 0 \\ &\Rightarrow x=0, \quad x+2y = \pi \quad \text{--- (ii)} \end{aligned}$$

On solving eq (i) and (ii)

$$\begin{aligned} 2x+y &= 2\pi \\ 2x+y &= \pi \end{aligned}$$

$$y = \pi/3 \rightarrow \text{Put in eq (i), we get } 2x + \pi/3 = \pi \\ \Rightarrow x = \pi/3$$

Therefore, we have stationary points are $(0,0)$ & $(\pi/3, \pi/3)$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 2 \sin y \cos(2x+y)$$

$$\begin{aligned} s = \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x} \sin x \sin(x+y) \\ &= \sin x \cos(x+y) + \sin(x+y) \cos x \\ &= \sin(2x+y) \end{aligned}$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x+2y)$$

① At point $(\pi/3, \pi/3)$

$$r = -\sqrt{3}$$

$$s = -\sqrt{3}/2$$

$$t = -\sqrt{3}$$

$$\bullet rt - s^2 = 3 - \frac{3}{4} > 0$$

$$\bullet r < 0$$

$\therefore u$ has maxima at $(\pi/3, \pi/3)$

$$[u]_{\max} = \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{8} //$$

⑥ At $(0, 0)$

$$r = 0$$

$$s = 0$$

$$t = 0$$

$$rt - s^2 = 0$$

Case unidentified, further investigation is needed.

Solⁿ (V) $u = x^3 y^2 (1 - x - y)$

$$\bullet \frac{\partial u}{\partial x} = 0, \quad x^3 y^2 (-1) + (1 - x - y) 3x^2 y^2 = 0$$

$$\Rightarrow x^2 y^2 (3 - 3x - 3y - x) = 0$$

$$\Rightarrow x^2 y^2 (3 - 4x - 3y) = 0$$

$$\Rightarrow x = 0, y = 0, 3 - 4x - 3y = 0 \quad \text{--- (i)}$$

$$\bullet \frac{\partial u}{\partial y} = 0, \quad x^3 y^2 (-1) + (1 - x - y) 2x^3 y = 0$$

$$\Rightarrow x^3 y (2 - 2x - 2y - y) = 0$$

$$\Rightarrow y x^3 (2 - 2x - 3y) = 0$$

$$\Rightarrow y = 0, x = 0, (2 - 2x - 3y) = 0 \quad \text{--- (ii)}$$

$$\Rightarrow -2x = 3y - 2$$

$$\Rightarrow -4x = 6y - 4 \rightarrow \text{put in (i)}$$

$$\Rightarrow 3 + 6y - 4 - 3y = 0$$

$$\Rightarrow y = \frac{1}{3} \rightarrow \text{put in (ii)}$$

$$\Rightarrow 2 - 2x - \frac{3}{3} = 0$$

$$\Rightarrow x = \frac{1}{2}$$

We have, stationary points are $(0, 0), (\frac{1}{2}, \frac{1}{3})$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = x^2 y^2 (-4) + (3 - 4x - 3y) 2x y^2$$

$$= x y^2 (6 - 8x - 6y - 4x) = x y^2 (6 - 12x - 6y)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} [y x^3 (2 - 2x - 3y)] = (2 - 2x - 3y) 3x^2 y + x^3 y (-2)$$

$$= x^2 y (6 - 6x - 3y - 2x) = x^2 y (6 - 8x - 3y)$$

$$\begin{aligned}
 t &= \frac{\partial^2 u}{\partial y^2} = yx^3(-3) + (2-2x-3y)x^3 \\
 &= x^3[2-2x-3y-3y] \\
 &= x^3(2-2x-6y)
 \end{aligned}$$

⊙ At point $(\frac{1}{2}, \frac{1}{3})$

$$r = \frac{1}{2}(\frac{1}{3})^2(6 - \frac{12}{2} - \frac{6}{3}) = \frac{1}{18}(6-6-2) = -\frac{1}{9}$$

$$s = (\frac{1}{2})^2(\frac{1}{3})(6 - \frac{8}{2} - \frac{9}{3}) = \frac{1}{12}(6-4-3) = -\frac{1}{12}$$

$$t = (\frac{1}{2})^3(2 - \frac{2}{2} - \frac{6}{3}) = \frac{1}{8}(2-1-2) = -\frac{1}{8}$$

$$\bullet \quad rt - s^2 = (-\frac{1}{9})(-\frac{1}{8}) - (\frac{1}{12})^2 = \frac{1}{72} - \frac{1}{144} > 0$$

$$\bullet \quad r < 0$$

∴ u has maxima at $(\frac{1}{2}, \frac{1}{3})$

$$\underline{\underline{[U]_{\max}}} = (\frac{1}{2})^3(\frac{1}{3})^2(1 - \frac{1}{2} - \frac{1}{3}) = \frac{1}{8}(\frac{1}{9})(\frac{1}{6}) = \frac{1}{432}$$

⊙ At point $(0, 0)$

$$r = 0$$

$$rt - s^2 = 0$$

$$s = 0$$

$$t = 0$$

Case unidentified, further investigation is needed.

Soln (VI)

$$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\bullet \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \quad \dots \textcircled{i}$$

$$\bullet \quad \frac{\partial u}{\partial y} = 0, \quad 4y^3 - 4y + 4x = 0 \quad \dots \textcircled{ii}$$

$$\text{On } \textcircled{i} + \textcircled{ii}$$

$$4x^3 + 4y^3 = 0$$

$$\Rightarrow x^3 + y^3 = 0$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$$\Rightarrow \boxed{x = -y} \rightarrow \text{put in ①}$$

$$\rightarrow -4y^3 - 4(-y) + 4y = 0$$

$$\Rightarrow y^3 - y - y = 0$$

$$\Rightarrow y(y^2 - 2) = 0$$

$$\Rightarrow y = 0, y^2 - 2 = 0$$

$$\Rightarrow y = \pm\sqrt{2}$$

$$\because x = -y$$

$$\because x = 0, x = -\sqrt{2}, x = +\sqrt{2}$$

We have stationary points are

$$(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$$

Now

$$r = \frac{\partial^2 u}{\partial x^2} = 4(3x^2 - 1)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 4(1)$$

$$t = \frac{\partial^2 u}{\partial y^2} = 4(3y^2 - 1)$$

① At point $(0, 0)$

$$r = -4$$

$$s = 4$$

$$t = -4$$

$$\bullet rt - s^2 = 4(4 - 4) = 0$$

$$\bullet r < 0$$

Case unidentified, further investigation is needed

① At $(\sqrt{2}, -\sqrt{2})$

$$x = \sqrt{3 \times 2 - 1} = (5)^4$$

$$y = 1 \times 4$$

$$z = \sqrt{3 \times 2 - 1} = (5)^4$$

$$\bullet x^2 - y^2 = (25 - 1) \times 4 > 0$$

$$x > 0$$

$\therefore u$ has minima at $(\sqrt{2}, -\sqrt{2})$

$$\begin{aligned} u_{\min} &= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 \\ &= 4 + 4 - 4 - 8 - 4 \end{aligned}$$

$$u_{\min} = -8$$

② At $(-\sqrt{2}, \sqrt{2})$

$$x = \sqrt{3 \times 2 - 1} = 5 \times 4$$

$$y = 1 \times 4$$

$$z = \sqrt{3 \times 2 - 1} = 5 \times 4$$

$$\bullet x^2 - y^2 = (25 - 1) \times 4 > 0 \quad x < 0$$

$\therefore u$ has also minima at $(-\sqrt{2}, \sqrt{2})$

$$x = \sqrt{3 \times 2 - 1} = 5 \times 4 \text{ and } \boxed{u_{\min} = -8}$$

Taylor's Series for Two Variables

$$f(x,y) = f(0,0) + \frac{1}{1!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f(0,0) + \frac{1}{2!} \left(x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} \right) f(0,0) \\ + \frac{1}{3!} \left(x^3 \frac{\partial^3}{\partial x^3} + 3x^2y \frac{\partial^3}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3}{\partial x \partial y^2} + y^3 \frac{\partial^3}{\partial y^3} \right) f(0,0) + \dots$$

Ex:- Expand e^{x+y} by Taylor's series, at $(0,0)$

Soln:- $f = f(x,y) = e^{x+y} \rightarrow f(0,0) = 1$

$$\frac{\partial f}{\partial x} = e^{x+y} \rightarrow \frac{\partial f(0,0)}{\partial x} = 1$$

$$\frac{\partial f}{\partial y} = e^{x+y} \rightarrow \frac{\partial f(0,0)}{\partial y} = 1$$

$$\frac{\partial f}{\partial x \partial y} = e^{x+y} \rightarrow \frac{\partial f(0,0)}{\partial x \partial y} = 1$$

$$\frac{\partial f}{\partial x^2} = e^{x+y} \rightarrow \frac{\partial f(0,0)}{\partial x^2} = 1$$

$$\frac{\partial f}{\partial y^2} = e^{x+y} \rightarrow \frac{\partial f(0,0)}{\partial y^2} = 1$$

Now

By ~~Taylor's Series~~

$$f(x,y) = f(0,0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f(0,0) + \left(x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} \right) f(0,0) + \dots$$

$$\Rightarrow e^{x+y} = 1 + (x \cdot 1 + y \cdot 1) + [x^2(1) + 2xy(1) + y^2(1)] + \dots$$

$$\Rightarrow e^{x+y} = 1 + (x+y) + (x+y)^2 + \dots //$$

Q Expand i) $\cos(x+y)$ by Taylor's series.
 ii) $e^x \log(1+y)$

Solⁿ: (i) Let $f(x, y) = \cos(x+y) \rightarrow f(0,0) = 1$

$$\frac{\partial f}{\partial x} = -\sin(x+y) \rightarrow \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = -\sin(x+y) \rightarrow \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = +\cos(x+y) \rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = -\cos(x+y) \rightarrow \frac{\partial^2 f}{\partial y^2}(0,0) = -1$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\cos(x+y) \rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = -1$$

By Taylor's series.

$$f(x, y) = \cos(x+y) = 1 + [x(0) + y(0)] + \frac{1}{2!}[x^2(-1) + 2xy(-1) + y^2(-1)] + \dots$$

$$\Rightarrow \cos(x+y) = 1 + \frac{x^2}{2} - xy - \frac{y^2}{2} + \dots //$$

$$\text{OR } \cos(x+y) = 1 - \frac{(x+y)^2}{2!} //$$

Solⁿ (ii) Let $f(x, y) = e^x \log(1+y) \rightarrow f(0,0) = 1$

$$\frac{\partial f}{\partial x} = e^x \log(1+y) \rightarrow \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{1+y} \rightarrow \frac{\partial f}{\partial y}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \log(1+y) \rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-e^x}{(1+y)^2} \rightarrow \frac{\partial^2 f_{(0,0)}}{\partial y^2} = -1$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^x}{1+y} \rightarrow \frac{\partial^2 f_{(0,0)}}{\partial x \partial y} = 1$$

By Taylor's Series

$$e^x \log(1+y) = 1 + (x+y) + (x^2 + 2xy - y^2)/2 \sim //$$

Rolle's Theorem:-

Let $f(x)$ be a function of variable x such that

- i) $f(x)$ is continuous in a closed interval $[a, b]$.
- ii) $f(x)$ is differentiable at every point in open interval (a, b) . And,
- iii) $f(a) = f(b)$ Then there is at least one point $x=c$, such that

$$f'(c) = 0$$

NOTE:-

- i) $[a, b]$; i.e. $a \leq x \leq b$
- ii) (a, b) ; i.e. $a < x < b$
- iii) $x=c$ is in (a, b)
i.e. $a < c < b$

iv) To find interval for which take $f(x) = 0$

v) Every polynomial is continuous in $[a, b]$ and differentiable in (a, b) at every point.

vi) Every differentiable is continuous but every continuous is not differentiable always.

Ex: 1 $\rightarrow$ Verify Rolle's Theorem for $f(x) = x^3 - 6x^2 + 11x - 6$

Solution: For interval take $f(x) = 0$

$$\Rightarrow f(x) = x^3 - 6x^2 + 11x - 6 = 0 \quad \Rightarrow f'(x) = 3x^2 - 12x + 11$$

$$\Rightarrow (x-1)(x^2 - 5x + 6) = 0$$

$$\Rightarrow (x-1)(x-3)(x-2) = 0$$

$$\Rightarrow x = 1, 3, 2$$

$\therefore$ Interval $[1, 3]$

Now, Given $f(x)$ is polynomial So,

i) $f(x)$ is continuous in $[1, 3]$.

ii) $f(x)$ is diffⁿ in $(1, 3)$

$$\text{Now, } f(1) = f(3) = 0 \quad | \quad f(2) = 0$$

$$\text{iii) } \Rightarrow f(a) = f(b)$$

Let $x=c$ in $(1, 3)$. Such that

$$f'(c) = 0$$

$$\Rightarrow 3c^2 - 12c + 11 = 0$$

$$\Rightarrow c = 2 \pm \frac{1}{\sqrt{3}}$$

$$= 2 + \frac{1}{\sqrt{3}}, 2 - \frac{1}{\sqrt{3}}$$

$$\in (1, 3), \in (1, 3)$$

Hence, Rolle's Theorem is verified for

$$c = 2 \pm \frac{1}{\sqrt{3}} //$$

Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x + 2$

Soln. To Find Interval, $f(x) = 0$

$$\Rightarrow f(x) = 2x^3 + x^2 - 4x + 2 = 0 \Rightarrow f'(x) = 6x^2 + 2x - 4$$

$$\Rightarrow (x + \frac{1}{2})(2x^2 - 4) = 0$$

$$\Rightarrow 2(x + \frac{1}{2})(x + \sqrt{2})(x - \sqrt{2}) = 0$$

$$\Rightarrow x = -\frac{1}{2}, -\sqrt{2}, +\sqrt{2}$$

$\therefore$ Interval $(-\frac{1}{2}, \sqrt{2})$

$\therefore$ Given $f(x)$ is polynomial

$\therefore$ i) $f(x)$ is continuous in $(-\frac{1}{2}, \sqrt{2})$

ii) $f(x)$ is differentiable in $(-\frac{1}{2}, \sqrt{2})$

$$\text{Now, } f(-\frac{1}{2}) = 0$$

$$f(\sqrt{2}) = 0$$

$$\text{iii) } f(b) = f(a) = 0$$

Let $x = c$ in $(-\frac{1}{2}, \sqrt{2})$ such that

$$f'(c) = 0$$

$$\Rightarrow 6c^2 + 2c - 4 = 0$$

$$\Rightarrow 3c^2 + c - 2 = 0$$

$$\Rightarrow 3c^2 + 3c - 2c - 2 = 0$$

$$\Rightarrow 3c(c+1) - 2(c+1) = 0$$

$$\Rightarrow (3c-2)(c+1) = 0$$

$$\Rightarrow c = \frac{2}{3}, -1$$

$$c \in (\frac{1}{2}, \sqrt{2}), \notin (-\frac{1}{2}, \sqrt{2})$$

Hence, Rolle's theorem is verified for $c = \frac{2}{3} //$

Lagrange's mean Value theorem -
(1st mean value theorem)

Let $f(x)$ = function in variable x such that

i) $f(x)$ is continuous in ~~every~~ closed interval $[a, b]$

ii) $f(x)$ is differentiable in open interval (a, b)

Then there is at least one point $x = c$ in (a, b) .

Such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$a, b \in \mathbb{R}$$

Problem Verify for Lagrange mean value theorem for

① $f(x) = (x-3)(x-6)(x-9)$ in $[3, 5]$

Soln- $f(x) = x^3 - 18x^2 + 99x - 162$

$$f'(x) = 3x^2 - 36x + 99$$

$$f'(c) = 3c^2 - 36c + 99$$

∵ $f(x)$ is polynomial so it is

i) Continuous in $[3, 5]$

ii) Differentiable in $(3, 5)$

∃ $x = c \in (3, 5)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{--- (1)}$$

$$f(a) = f(3) = 0$$

$$f(b) = f(5) = 8$$

from (1)

$$\frac{8 - 0}{5 - 3} = 3c^2 - 36c + 99$$

$$\Rightarrow 3c^2 - 36c + 95 = 0$$

$$\Rightarrow c = \frac{6 \pm \sqrt{39}}{3}$$

$$= 6 + \frac{\sqrt{39}}{3}, 6 - \frac{\sqrt{39}}{3}$$

$$\notin (3, 5) \in (3, 5)$$

Hence, Lagrange theorem is verified for $c = 6 - \frac{\sqrt{39}}{3}$
~~Lagrange Theorem,~~

Q I) Verify Rolle's theorem for

i) $f(x) = x^3 - 12x$; $0 \leq x \leq 2\sqrt{3}$

ii) $f(x) = x^3 - 4x$

iii) $f(x) = 1 - (x-3)^{2/3}$

iv) $f(x) = (x+3)x e^{-1/2x}$; $[-3, 0]$

Soln

i) $f(x) = x^3 - 12x \Rightarrow f'(x) = 3x^2 - 12$

Interval $[0, 2\sqrt{3}] \Rightarrow f'(c) = 3c^2 - 12$

∵ $f(x)$ is polynomial

∴ i) $f(x)$ is continuous in $[0, 2\sqrt{3}]$

ii) $f(x)$ is differentiable in $(0, 2\sqrt{3})$

Also

$$f(a) = f(0) = 0$$

$$f(b) = f(2\sqrt{3}) = (2\sqrt{3})^3 - 12(2\sqrt{3}) = 8 \times 3\sqrt{3} - 24\sqrt{3} = 0$$

$$\Rightarrow f(a) = f(b)$$

$\exists x=c \in (a,b)$ such that

$$f'(c) = 0$$

$$\Rightarrow 3c^2 - 4 = 0$$

$$\Rightarrow c^2 = 4/3 = 4$$

$$\Rightarrow c = +2, -2$$

$$\in (0, 2\sqrt{3}), \notin (0, 2\sqrt{3})$$

Hence, Rolle's theorem

is verified for

$$c = 2$$

Soln

$$f(x) = x^3 - 4x \Rightarrow f'(x) = 3x^2 - 4$$

$$\text{For Interval } f(x) = 0 \Rightarrow f'(c) = 3c^2 - 4$$

$$\Rightarrow x^3 - 4x = 0$$

$$\Rightarrow x(x^2 - 4) = 0$$

$$\Rightarrow x = 0, -2, +2$$

$$\therefore \text{Interval } [-2, 2]$$

$\because f(x)$ is polynomial.

$\therefore$ i) It is continuous $[0, 2]$

ii) Diffⁿ in $(0, 2)$

Also

$$f(a) = f(0) = 0$$

$$f(b) = f(2) = 0$$

$$\Rightarrow f(a) = f(b)$$

$\exists x=c \in (a,b)$ such that

$$f'(c) = 0$$

$$3c^2 - 4 = 0$$

$$\Rightarrow c = \pm \frac{2}{\sqrt{3}}$$

$$\Rightarrow c = +\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$$

$$\in (-2, 2) \quad (-2, 2)$$

Hence, Rolle's theorem is

verified for both

$$c = \frac{2}{\sqrt{3}} \text{ and } -\frac{2}{\sqrt{3}} //$$

Soln

$$f(x) = 1 - (x-3)^{2/3}$$

$$\text{for Interval } f(x) = 0$$

$$\Rightarrow 1 - (x-3)^{2/3} = 0$$

$$\Rightarrow (x-3)^{2/3} = 1$$

$$\Rightarrow (x-3)^2 = 1$$

$$\Rightarrow x-3 = \pm 1 \Rightarrow x = 4, 2$$

24/12/24

we get Interval $[2, 4]$

Now
$$f'(x) = 0 - \frac{2}{3}(x-3)^{-\frac{2}{3}}$$
$$= -\frac{2}{3}(x-3)^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = -\frac{2}{3} \frac{1}{(x-3)^{\frac{1}{3}}}$$

But at $x=3$ in $(2, 4)$

$$f'(3) = \infty \quad \therefore f'(x) \text{ does not exist.}$$

Hence, $f(x)$ is not differentiable in $(2, 4)$

Rolle's theorem is not verified //

Solⁿ iv) $f(x) = (x+3)x e^{-\frac{1}{2}x}$

Interval $[-3, 0]$

$$\Rightarrow f'(x) = e^{-\frac{1}{2}x} [x^2 + 3x]$$

$$\Rightarrow f'(x) = e^{-\frac{x}{2}} [2x+3] + (x^2+3x) e^{-\frac{x}{2}} \left(-\frac{1}{2}\right)$$

$$= e^{-\frac{x}{2}} \left[2x+3 - \frac{x^2}{2} - \frac{3x}{2} \right]$$

$$= \frac{e^{-\frac{x}{2}}}{2} [-x^2 + x + 6]$$

$$\Rightarrow f'(c) = \frac{e^{-\frac{c}{2}}}{2} [-c^2 + c + 6]$$

$$\begin{array}{l} f(-3) = 0 = f(a) \\ f(0) = 0 = f(b) \end{array}$$

Now $\because f(a) = f(b)$

$\therefore \exists x=c \in (a, b)$ such
that $f'(c) = 0$

$$\Rightarrow -\frac{e^{-\frac{c}{2}}}{2} [c^2 - c - 6] = 0$$

$$\frac{e^{-\frac{c}{2}}}{2} \neq 0 \quad \forall x \in \mathbb{R}$$

$$\therefore c^2 - c - 6 = 0$$

$$\Rightarrow c^2 - 3c + 2c - 6 = 0$$

$$\Rightarrow (c-3)(c+2) = 0$$

$$\Rightarrow c = 3, -2$$

$$\notin (-3, 0), \in (-3, 0)$$

Hence, Rolle's theorem
is verified for

$$c = (-2) //$$

II Verify Lagrange's mean Value Theorem-

i) $f(x) = \sqrt{x^2-4}$, $[2, 4]$

ii) $f(x) = x^3$, $[-2, 2]$

iii) $f(x) = \log x$, $[1, e]$

iv) $f(x) = 2x^2 - 7x + 10$, $[2, 5]$

Solⁿ i) $f(x) = \sqrt{x^2-4} = (x^2-4)^{1/2} \Rightarrow f'(x) = \frac{1}{2} (x^2-4)^{-1/2} \times 2x$

Interval $[2, 4]$

$\Rightarrow f'(x) = x (x^2-4)^{-1/2}$

$\because f'(x)$ is differentiable

$\Rightarrow f'(c) = \frac{c}{\sqrt{c^2-4}}$

in $(2, 4)$

It is continuous in $[2, 4]$

$f(a) = f(2) = 0$

$f(b) = f(4) = 2\sqrt{3}$

By Lagrange's mean value

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{2\sqrt{3} - 0}{4 - 2} = \frac{c}{\sqrt{c^2-4}}$$

$$\Rightarrow 2\sqrt{3} \sqrt{c^2-4} = 2c$$

$$\Rightarrow 3(c^2-4) = c^2$$

$$\Rightarrow 2c^2 = 12$$

$$\Rightarrow c = \pm\sqrt{6}$$

$$\Rightarrow c = +\sqrt{6}, -\sqrt{6}$$

$\in (2, 4), \notin (2, 4)$

Hence, Lagrange's theorem is verified for $c = +\sqrt{6} \in (2, 4)$

Solⁿ ii) $f(x) = x^3$

Interval $[-2, 2]$

$$\Rightarrow f'(x) = 3x^2$$

$$\Rightarrow f'(c) = 3c^2$$

$\because f(x)$ is poly^m

$\therefore$ i) It is continuous in $[-2, 2]$

ii) It is differentiable in $(-2, 2)$

Now

$$f(a) = f(-2) = -8 \quad | \quad f(b) = f(2) = 8$$

By Lagrange's theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{+8-(-8)}{2-(-2)} = 3c^2$$

$$\Rightarrow \frac{16}{4} = 3c^2$$

$$\Rightarrow 4/3 = c^2 \Rightarrow c = \pm \frac{2}{\sqrt{3}} \in [2,2], \in [-2,2]$$

Hence, Lagrange theorem is verified for both $c = \pm \frac{2}{\sqrt{3}}$

Soln i) $f(x) = \log(x)$. interval $[1, e]$ $\therefore 2 < e < 3$

$$f'(x) = \frac{1}{x}$$

$$\Rightarrow f'(c) = \frac{1}{c}$$

$\therefore f(x)$ is continuous in $[1, e]$

$f(x)$ is differentiable in $(1, e)$

Now

$$f(a) = f(1) = \log 1 = 0$$

$$f(b) = f(e) = \log e = 1$$

By Lagrange's theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{1-(0)}{e-1} = \frac{1}{c}$$

$$\Rightarrow c = e-1 \in (1, e)$$

Hence, Lagrange theorem is verified for $c = e-1$ in $(1, e)$

Soln iv) $f(x) = 2x^2 - 7x + 10$. interval $[2, 5]$

$$f'(x) = 4x - 7$$

$$\Rightarrow f'(c) = 4c - 7$$

$\therefore f(x)$ is polynomial

$\therefore f$ is continuous in $[2, 5]$

$\therefore f$ is differentiable in $[2, 5]$

Now

$$f(a) = f(2) = 2 \times 2^2 - 7 \times 2 + 10 = 4$$

$$f(b) = f(5) = 2 \times 5^2 - 7 \times 5 + 10 = 25$$

By Lagrange's theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{25 - 4}{5 - 2} = 4c - 7$$

$$\Rightarrow \frac{21}{3} + 7 = 4c$$

$$\Rightarrow 14 = 4c$$

$$\Rightarrow 3.5 = c \in (2, 5)$$

Hence, Lagrange theorem

is verified for $c = \frac{14}{4}$

$\in (2, 5)$

Unit : 2

Integral Calculus

Integral as limit of Sum

If $f(x)=y$ is a continuous function in x defined in $[a, b]$ $a, b \in \mathbb{R}$

Then,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

$$\text{Where } nh = b-a$$

$$\text{or } n = \frac{b-a}{h}$$

$$\text{or } h = \frac{b-a}{n}$$

Proof:

Let $[a, b]$ be divided into n equal parts of width $h = \frac{b-a}{n}$, Then,

$\int_a^b f(x) dx$ gives area bounded by $y=f(x)$, $x=a$, $x=b$ and x -axis.

$$\int_a^b f(x) dx = \begin{array}{c} \text{Area under } y=f(x) \\ \text{from } x=a \text{ to } x=b \end{array} = \begin{array}{c} \text{Area of } n \text{ rectangles} \\ \text{under condition as } h \rightarrow 0 \end{array}$$

$$= \lim_{h \rightarrow 0} [h f(a) + h f(a+h) + h f(a+2h) + \dots + h f(a+(n-1)h)]$$

$$\Rightarrow \boxed{\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]}$$

Some formulae to solve limits:-

$$(i) 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$(ii) 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(iii) 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\text{G.P.} \\ \text{iv)} \cdot a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} : r > 1$$

$$\cdot a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} ; r < 1$$

$$\text{v)} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1$$

$$\text{vi)} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{x}{e^x - 1} = 1$$

$$\text{vii)} \lim_{\theta \rightarrow 0} \cos \theta = 1$$

$$\begin{aligned} \text{viii)} & \sin a + \sin(a+h) + \sin(a+2h) + \dots + \sin\{a+(n-1)h\} \\ &= \frac{\sin\left\{a + \left(\frac{n-1}{2}\right)h\right\} \cdot \sin \frac{nh}{2}}{\sin h/2} \end{aligned}$$

$$\begin{aligned} \text{ix)} & \cos a + \cos(a+h) + \cos(a+2h) + \dots + \cos\{a+(n-1)h\} \\ &= \frac{\cos\left\{a + \left(\frac{n-1}{2}\right)h\right\} \sin \frac{nh}{2}}{\sin h/2} \end{aligned}$$

Q Evaluate $\int_a^b x dx$ by using limit of sum.

Solⁿ Let $f(x) = x$

$$f(a) = a, f(a+h) = a+h, f(a+2h) = a+2h \dots + f(a+(n-1)h) \\ = (a + (n-1)h)$$

We have

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h \left[f(a) + f(a+h) + \dots + f(a+(n-1)h) \right]$$

$$\text{where } nh = b-a / n = \frac{b-a}{h} \quad | \quad h = \frac{b-a}{n}$$

$$\int_a^b x dx = \lim_{h \rightarrow 0} h \left[a + (a+h) + (a+2h) + \dots + \{a+(n-1)h\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[na + h(1+2+\dots+n) \right]$$

$$= \lim_{h \rightarrow 0} h \left[na + h \frac{n(n+1)}{2} \right]$$

$$= \lim_{h \rightarrow 0} \left[nh(a) + nh \frac{(nh-h)}{2} \right]$$

$$= \lim_{h \rightarrow 0} \left[(b-a)a + (b-a) \frac{(b-a)-h}{2} \right]$$

$$= \lim_{h \rightarrow 0} \left[(b-a) \left[\frac{2a+b-a+h}{2} \right] \right]$$

$$= \frac{b^2 - a^2}{2}$$

$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} //$$

Q Evaluate $\int_a^b x^2 dx$ by as a limit of sum.

Soln Let $y = f(x) = x^2$

$$f(a) = a^2, f(a+h) = (a+h)^2, f(a+2h) = (a+2h)^2, \dots$$

$$f[a+(n-1)h] = [a+(n-1)h]^2$$

$$\therefore \int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + \dots + f[a+(n-1)h]]$$

$$\therefore \int_a^b x^2 dx = \lim_{h \rightarrow 0} h [a^2 + (a+h)^2 + \dots + [a+(n-1)h]^2]$$

$$= \lim_{h \rightarrow 0} h [a^2 + (a^2 + 2ah + h^2) + \dots + [a^2 + 2ah(n-1) + (n-1)^2 h^2]]$$

$$= \lim_{h \rightarrow 0} h [na^2 + h^2 \{1^2 + 2^2 + 3^2 + \dots + (n-1)^2\} + 2ah \{1 + 2 + 3 + \dots + (n-1)\}]$$

$$= \lim_{h \rightarrow 0} h [na^2 + h^2 \left\{ \frac{n(n-1)(2n-1)}{6} \right\} + 2ah \left\{ \frac{n(n-1)}{2} \right\}]$$

$$= \lim_{h \rightarrow 0} \left[nh a^2 + \frac{nh(nh-h)(2nh-h)}{6} + a \{nh(nh-h)\} \right]$$

$$\text{Put } nh = b-a, h \rightarrow 0$$

$$= \left[\frac{(b-a)}{1} a^2 + \frac{(b-a)(b-a)^2(b-a)}{6} + a(b-a)(b-a) \right]$$

$$= \left[\frac{a^2}{1} + \frac{a^2 + b^2 - 2ab}{3} + \frac{ab - a^2}{1} \right] (b-a)$$

$$= (b-a) \left[\frac{3a^2 + a^2 + b^2 - 2ab + 3ab - 3a^2}{3} \right]$$

$$\Rightarrow \int_a^b x^2 dx = \left(\frac{b-a}{3}\right)(b^2+ba+a^2)$$

$$\Rightarrow \int_a^b x^3 dx = \frac{b^3}{3} - \frac{a^3}{3} //$$

¶ Evaluate $\int_a^b x^3 dx$ by as a limit of Sum.

Sol Let $f(x) = x^3$;

$$f(a) = a^3, f(a+h) = (a+h)^3 + f(a+2h) = (a+2h)^3 + \dots$$

Now

$$\int_a^b x^3 dx = \lim_{h \rightarrow 0} h \left[a^3 + (a+h)^3 + \dots + \{a+(n-1)h\}^3 \right]$$

$$= \lim_{h \rightarrow 0} h \left[a^3 + (a^3 + 3a^2h + 3ah^2 + h^3) + \dots + \{a^3 + 3a^2(n-1)h + 3a(n-1)^2h^2 + (n-1)^3h^3\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[na^3 + h^3 \{1^3 + 2^3 + 3^3 + \dots + (n-1)^3\} + 3a^2h \{1 + 2 + 3 + \dots + (n-1)\} + 3ah^2 \{1^2 + 2^2 + 3^2 + \dots + (n-1)^2\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[na^3 + h^3 \left\{ \frac{n(n-1)}{2} \right\}^2 + 3a^2h \left\{ \frac{n(n-1)}{2} \right\} + 3ah^2 \left\{ \frac{n(n-1)(2n-1)}{6} \right\} \right]$$

$$\begin{aligned}
 \int_a^b x^3 dx &= \lim_{h \rightarrow 0} \left[nha^3 + \left\{ \frac{(nh)(nh-h)}{2} \right\}^2 + 3a^2 \left\{ \frac{nh(nh-h)}{2} \right\} \right. \\
 &\quad \left. + 3a \left\{ \frac{nh(nh-h)(nh-h)}{2} \right\} \right] \\
 &= \lim_{h \rightarrow 0} \left[(b-a)a^3 + \left[\frac{(b-a)(b-a-h)}{2} \right]^2 + \left[\frac{3a^2(b-a)(b-a-h)}{2} \right] \right. \\
 &\quad \left. + 3a \left[\frac{(b-a)\{(b-a)-h\}\{2(b-a)-h\}}{6} \right] \right] \\
 &= (b-a)a^3 + \left[\frac{(b-a)(b-a)}{2} \right]^2 + \left[\frac{3a^2(b-a)(b-a)}{2} \right] \\
 &\quad + \left[\frac{3a(b-a)(b-a)(b-a)}{2} \right]
 \end{aligned}$$

$$= (b-a) \left[a^3 + \frac{(b-a)^3}{4} + \frac{3a^2(b-a)}{2} + a(b-a)^2 \right]$$

$$= (b-a) \left[a^3 + \frac{b^3 + 3ba^2 - 3ab^2 - a^3}{4} + \frac{3a^2b - 3a^3}{2} + ab^2 + a^3 - 2a^2b \right]$$

$$= \frac{b-a}{4} \left[\underline{4a^3} + \underline{b^3} + \underline{3ba^2} - \underline{3ab^2} - \underline{a^3} + \underline{6a^2b} - \underline{6a^3} + \underline{4ab^2} + \underline{4a^3} - \underline{8a^2b} \right]$$

$$= \frac{b-a}{4} \left[a^3 + b^3 + 1ba^2 + 1a \cdot b^2 \right]$$

$$= \frac{1}{4} \left[\cancel{a^3b} + b^4 + \cancel{1b^2a^2} + \cancel{1a \cdot b^3} - \cancel{a^4} - \cancel{a^3b} - \cancel{ba^3} + \cancel{1a^2b^2} \right]$$

$$\int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4} //$$

Q Evaluate $\int_a^b e^x dx$ as limit of sum.

$$\underline{\text{Soln}} \quad \int_a^b e^x dx = \lim_{h \rightarrow 0} h \left[e^a + e^{(a+h)} + e^{(a+2h)} + \dots + e^{[a+(n-1)h]} \right]$$

$$= \lim_{h \rightarrow 0} h \left[e^a \{ 1 + e^h + e^{2h} + \dots + e^{(n-1)h} \} \right]$$

$$= \lim_{h \rightarrow 0} h e^a \left\{ \frac{e^{nh} - 1}{e^h - 1} \right\}$$

$$= \lim_{h \rightarrow 0} \left[\frac{h}{e^h - 1} \right] \{ e^a (e^{nh} - 1) \}$$

$$= 1 \cdot e^a (e^{b-a} - 1)$$

$$\int_a^b e^x dx = e^b - e^a //$$

Q Evaluate $\int_a^b e^{kx} dx$ as limit of sum.

$$\underline{\text{Soln}} \quad \int_a^b e^{kx} dx = \lim_{h \rightarrow 0} h \left[e^{ka} + e^{k(a+h)} + e^{k(a+2h)} + \dots + e^{k[a+(n-1)h]} \right]$$

$$= \lim_{h \rightarrow 0} h e^{ka} \left[1 + e^{kh} + e^{2kh} + \dots + e^{(n-1)kh} \right]$$

$$= \lim_{h \rightarrow 0} h e^{ka} \left[\frac{e^{knh} - 1}{e^{kh} - 1} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{kh}{e^{kh} - 1} \right] \left\{ \frac{1}{k} \{ e^{ka} (e^{knh} - 1) \} \right\}$$

$$= \frac{1}{k} \{ e^{ka} (e^{k(b-a)} - 1) \}$$

$$\int_a^b e^{kx} dx = \frac{1}{k} [e^{kb} - e^{ka}] //$$

≠ $\int_a^b \sin(x) dx$ as the limit of sum. Find $\int_0^{\pi/2} \sin(x) dx$

Solⁿ

$$\int_a^b \sin(x) dx = \lim_{h \rightarrow 0} h \left[\sin(a) + \sin(a+h) + \sin(a+2h) + \dots + \sin[a+(n-1)h] \right]$$

$$\Rightarrow \int_a^b \sin(x) dx = \lim_{h \rightarrow 0} h \left[\frac{\sin\left\{a+(n-1)\frac{h}{2}\right\} \sin\left(\frac{nh}{2}\right)}{\sin h/2} \right]$$

$$= \lim_{h \rightarrow 0} h \left[\frac{\sin\left\{a+\frac{nh}{2}-\frac{h}{2}\right\} \sin\left(\frac{nh}{2}\right)}{\frac{\sin h/2 \times h/2}{h/2}} \right]$$

$$= \lim_{h \rightarrow 0} \frac{h/2}{h/2} \left[\frac{2 \sin\left\{a+\frac{b-a}{2}-\frac{h}{2}\right\} \sin\left(\frac{b-a}{2}\right)}{\frac{\sin h/2}{h/2}} \right]$$

$$= 2 \sin\left(\frac{b+a}{2}\right) \times \sin\left(\frac{b-a}{2}\right)$$

$$= \cos\left\{\frac{a+b}{2}-\frac{b-a}{2}\right\} - \cos\left\{\frac{a+b}{2}+\frac{b-a}{2}\right\}$$

$$\int_a^b \sin(x) dx = \cos a - \cos b //$$

Put $a=0, b=\pi/2$

$$\int_0^{\pi/2} \sin(x) dx = \cos 0 - \cos \pi/2 = 1 - 0 = 1 //$$

Q Evaluate $\int_a^b \cos(x) dx$, by as a limit of Sum.

Soln

$$\int_a^b \cos(x) dx = \lim_{h \rightarrow 0} \left[h \left\{ \cos a + \cos(a+h) + \cos(a+2h) + \dots + \cos\{a+(n-1)h\} \right\} \right]$$
$$= \lim_{h \rightarrow 0} h \left[\frac{\cos\{a + (\frac{n-1}{2})h\} \sin(\frac{nh}{2})}{\sin h/2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{h/2}{\sin h/2} \times 2 \cos\left(a + \frac{nh-h}{2}\right) \times \sin\left(\frac{nh}{2}\right)$$

$$= \lim_{h \rightarrow 0} \frac{h/2}{\sin h/2} \times 2 \cos\left(a + \frac{b-a-h}{2}\right) \times \sin \frac{b-a}{2}$$

$$= 1 \times 2 \cos\left(\frac{b+a}{2}\right) \sin\left(\frac{b-a}{2}\right)$$

$$\int_a^b \cos(x) dx = \sin b - \sin a \quad \left[\begin{array}{l} \sin C - \sin D \\ = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \end{array} \right]$$

Summation Series :-

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

$$= \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} f\{a+rh\}$$

Let $a=0$, $b=1$

$$\therefore nh = b-a$$

$$nh = 1-0$$

When $h \rightarrow 0$

$$n \rightarrow \infty$$

$$h = \frac{1}{n}$$

Now

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) \quad \therefore [h = \frac{1}{n}]$$

Working Rule -

i) Arrange each term in similar form according to problem.

ii) To find r th term in summation form.

iii) Make $\frac{1}{n}$ and $\frac{r}{n}$ in summation

iv) To find limit

$$\text{Lower limit} = \lim_{n \rightarrow \infty} \left(\frac{r}{n}\right) \rightarrow \text{lower value of } r$$

$$\text{Upper limit} = \lim_{n \rightarrow \infty} \left(\frac{r}{n}\right) \rightarrow \text{upper value of } r$$

Q Evaluate $\lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} \right]$

Let $p = \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} \right]$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n+r} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n(1+\frac{r}{n})}$$

$$L.L = \lim_{n \rightarrow \infty} \left(\frac{x}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \because (x=1)$$

$$= 0$$

$$U.L = \lim_{n \rightarrow \infty} \left(\frac{x}{n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{n} \right) \because [x=n]$$

$$= 1$$

P becomes

$$\int_0^1 \frac{1}{1+x} dx = \left[\log(1+x) \right]_0^1 = \log 2 - \log 1$$

$$\Rightarrow P = \log 2 //$$

Q Evaluate $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{\frac{1}{n}}$

Soln let $P = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{\frac{1}{n}}$

On taking log both side

$$\log P = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log\left(1 + \frac{1}{n}\right) + \log\left(1 + \frac{2}{n}\right) + \dots + \log\left(1 + \frac{n}{n}\right) \right]$$

$$\log P = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\log\left(1 + \frac{r}{n}\right) \right] \quad \dots \text{--- (1)}$$

$$L.L = \lim_{n \rightarrow \infty} \frac{x}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$$U.L = \lim_{n \rightarrow \infty} \frac{x}{n} = \lim_{n \rightarrow \infty} \frac{n}{n} = 1$$

eqn (1) becomes

$$\log P = \int_0^1 \log(1+x) dx = \left[(1+x) \log(1+x) - x \right]_0^1$$

$$\because \int \log x dx = x(\log x - 1)$$

$$\log P = 2[\log 2 - 1] - 1[\log 1 - 1]$$

$$\Rightarrow \log P = 2\log 2 - 2 + 1 = 2\log 2 - 1 = \log 4 - \log e$$

$$\Rightarrow \log P = \log \frac{4}{e}$$

$$\Rightarrow P = \frac{4}{e}$$

Q Evaluate $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^1 + \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} + \left(1 + \frac{1}{n}\right)^{\frac{1}{3}} + \dots + \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} \right]$

Soln let $P = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^1 + \left(1 + \frac{1}{n}\right)^{\frac{1}{2}} + \left(1 + \frac{1}{n}\right)^{\frac{1}{3}} + \dots + \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} \right]$

Taking log both side.

$$\Rightarrow \log P = \lim_{n \rightarrow \infty} \left[\log \left(1 + \frac{1}{n}\right) + \frac{1}{2} \log \left(1 + \frac{1}{n}\right) + \frac{1}{3} \log \left(1 + \frac{1}{n}\right) + \dots + \frac{1}{n} \log \left(1 + \frac{1}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{r} \log \left(1 + \frac{1}{n}\right)$$

$$\Rightarrow \log P = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{r} \cdot \frac{1}{n} \log \left(1 + \frac{r}{n}\right) \quad \text{--- (1)}$$

$$\text{L.O.L} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \because [r=1]$$

$$\text{U.L} = \lim_{n \rightarrow \infty} \frac{r}{n} = \lim_{n \rightarrow \infty} \frac{n}{n} = 1 \quad \because [r=n]$$

eqn (1) become

$$\log P = \int_0^1 \frac{1}{x} \log(1+x) dx$$

$$\begin{aligned}
 &= \int_0^1 \frac{1}{x} \log(1+x) dx \\
 &= \int_0^1 \frac{1}{x} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) dx \\
 &= \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right) dx \\
 &= \left[x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \dots \right]_0^1
 \end{aligned}$$

$$\Rightarrow \log P = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

$$\Rightarrow P = e^{\pi^2/12} \quad \underline{\text{Ans}}$$

Q Evaluate $\lim_{n \rightarrow \infty} \left\{ \frac{Ln}{n^n} \right\}^{1/n}$

Soln let $P = \lim_{n \rightarrow \infty} \left\{ \frac{1 \cdot 2 \cdot 3 \dots n}{n \cdot n \cdot n \dots n} \right\}^{1/n}$

$$P = \lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right\}^{1/n}$$

taking log both side

$$\log P = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \log\left(\frac{3}{n}\right) + \dots + \log\left(\frac{n}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \log\left(\frac{r}{n}\right)$$

$$\Rightarrow \log P = \int_0^1 \log x dx$$

$$\log P = \left[x(\log x - 1) \right]_0^1 = \log 1 - 1 - 0 = -1$$

$$\therefore P = e^{-1} = 1/e //$$

Q Evaluate $\lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3}{n^2} + \dots + \frac{n}{n^2} \sec^2 \frac{n}{n^2} \right]$

Solⁿ

Let $P = \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{2}{n^2} + \frac{3}{n^2} \sec^2 \frac{3}{n^2} + \dots + \frac{n}{n^2} \sec^2 \frac{n}{n^2} \right]$

$P = \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{r}{n} \sec^2 \frac{r^2}{n^2} \right] \quad \text{--- (1)}$

$= \int_0^1 x \sec^2 x^2 dx$

Put $x^2 = t$

$\Rightarrow 2x dx = dt$

$\Rightarrow x dx = \frac{1}{2} dt$

$x \rightarrow 0 \quad t \rightarrow 0$

$x \rightarrow 1 \quad t \rightarrow 1$

$\Rightarrow P = \frac{1}{2} \int_0^1 \sec^2 t dt = \frac{1}{2} [\tan t]_0^1 = \frac{1}{2} [\tan 1 - \tan 0]$

$\Rightarrow P = \frac{1}{2} \tan(1) //$

Q Evaluate $\lim_{n \rightarrow \infty} \left[\frac{n^2}{n^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{n^2}{(n+n)^3} \right]$

Solⁿ
Let $P = \lim_{n \rightarrow \infty} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{n^2}{(n+n)^3} \right]$

$P = \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{n^2}{(n+r)^3}$

$P = \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{n} \left[\frac{1}{\left(1 + \frac{r}{n}\right)^3} \right]$

L.L. = $\lim_{n \rightarrow \infty} \frac{0}{n} = 0$

U.P. = $\lim_{n \rightarrow \infty} \frac{n}{n} = 1$

$$P = \int_0^1 \frac{1}{(1+x)^3} dx = \left[-\frac{1}{2} \frac{1}{(1+x)^2} \right]_0^1 = -\frac{1}{2} \left[\frac{1}{4} - 1 \right] = \frac{3}{8} //$$

Q Evaluate $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right]$

Solⁿ: Let $P = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right]$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{n}} \left[\frac{1}{\sqrt{r}} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\frac{1}{\sqrt{r/n}} \right]$$

$$L \cdot L = \lim_{n \rightarrow \infty} \frac{0}{n} = 0$$

$$U \cdot P = \lim_{n \rightarrow \infty} \frac{n}{n} = 1$$

$$\Rightarrow P = \int_0^1 \frac{1}{\sqrt{x}} dx = \left[\frac{\sqrt{x}}{2} \right]_0^1 = 2 //$$

Q Evaluate 1. $\lim_{n \rightarrow \infty} n \left[\frac{1}{n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \dots + \frac{1}{n^2+(n-1)^2} \right]$

Solⁿ: Let $P = \lim_{n \rightarrow \infty} n \left[\frac{1}{n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \dots + \frac{1}{n^2+(n-1)^2} \right]$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} n \frac{1}{n^2+r^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} \left[\frac{1}{1 + \left(\frac{r}{n}\right)^2} \right]$$

$$L \cdot L = \lim_{n \rightarrow \infty} \frac{x}{n} = 0$$

$$U \cdot L = \lim_{n \rightarrow \infty} \frac{x}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{n-1}{n}$$

$$= 1$$

$$= \int_0^1 \frac{1}{1+x^2} dx = \left[\tan^{-1}(x) \right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4 - 0$$

$$= \pi/4 //$$

$$Q \quad \lim_{n \rightarrow \infty} \sum_{r=1}^{3n} \frac{n^2}{(3n+r)^3}$$

$$\text{Soln:} \quad \text{Given } \lim_{n \rightarrow \infty} \sum_{r=1}^{3n} \frac{n^2}{(3n+r)^3} = \sum_{r=1}^{3n} \frac{1}{n} \left[\frac{1}{(3+\frac{r}{n})^3} \right]$$

$$L.L = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0$$

$$U.P = \lim_{n \rightarrow \infty} \left(\frac{3n}{n} \right) = 3$$

$$\Rightarrow \int_0^3 \frac{1}{(3+x)^3} dx = \left[\frac{(3+x)^{-2}}{-2} \right]_0^3 = -\frac{1}{12} + \frac{1}{18} = \frac{3}{12} = \frac{1}{4} //$$

$$Q \quad \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{n^2 - r^2}}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\frac{1}{\sqrt{1 - (\frac{r}{n})^2}} \right] \quad \begin{matrix} L.L = 0 \\ U.L = 1 \end{matrix}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1 = \sin^{-1} 1 - \sin^{-1} 0 \\ = \pi/2 - 0 \\ = \pi/2 //$$

$$Q \quad \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n+r}{n^2+r^2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \left[\frac{1 + \frac{r}{n}}{1 + (\frac{r}{n})^2} \right]$$

$$= \int_0^1 \frac{1+x}{1+x^2} dx = \int_0^1 \left[\frac{2x}{2(1+x^2)} + \frac{1}{1+x^2} \right] dx \\ = \left[\frac{\log(1+x^2)}{2} + \tan^{-1} x \right]_0^1 \\ = \frac{\log 2}{2} + \frac{\pi}{4} //$$

Beta and Gamma Function

Beta function:-

If $m, n > 0$, Then Beta function β is defined as

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

NOTE

$$i) \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$ii) \beta(m+1, n) = \int_0^1 x^{m+1-1} (1-x)^{n-1} dx$$

$$iii) \beta(m, n+1) = \int_0^1 x^{m-1} (1-x)^{n+1-1} dx$$

Example:-

$$\begin{aligned} \beta(1/2, 3/2) &= \int_0^1 x^{1/2-1} (1-x)^{3/2-1} dx \\ &= \int_0^1 x^{-1/2} (1-x)^{1/2} dx \\ &= \int_0^1 \sqrt{\frac{1-x}{x}} dx // \end{aligned}$$

Properties of Beta function:-

Theorem-1 :- Beta function holds the symmetric law
ie, $\beta(m, n) = \beta(n, m)$

Proof:- By Defⁿ of Beta function.

$$\begin{aligned} \beta(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \because \left[\int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \\ &= \int_0^1 (1-x)^{m-1} \{1-(1-x)\}^{n-1} dx \\ &= \int_0^1 x^{n-1} (1-x)^{m-1} dx = \beta(n, m) \end{aligned}$$

Proved

Theorem - 2 :- $\beta(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{n+m}} dx$

Proof: By Definition of Beta function

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put $x = \frac{1}{1+y}$ or $(1+y) = \frac{1}{x}$

$$\Rightarrow dx = -\frac{1}{(1+y)^2} dy \quad \left| \begin{array}{l} \text{If } x \rightarrow 0 ; y \rightarrow \infty \\ x \rightarrow 1 ; y \rightarrow 0 \end{array} \right.$$

$$\therefore \beta(m, n) = \int_{\infty}^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \left[-\frac{1}{(1+y)^2}\right] dy$$

$$= - \int_{\infty}^0 \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\Rightarrow \beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\int f(x) dx = \int f(t) dt$$

$$\therefore \beta(m, n) = \beta(n, m)$$

$$\therefore \beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{n+m}} dx = \beta(n, m)$$

Proved

Theorem-3 :-

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1}(\theta) \cdot \cos^{2n-1}(\theta) d\theta$$

Proof:- By Defⁿ of $\beta(m, n)$

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Put } x = \sin^2 \theta$$

$$\Rightarrow dx = 2 \sin \theta \cos \theta d\theta$$

$$\left| \begin{array}{l} \text{If } x \rightarrow 0 ; \theta \rightarrow 0 \\ x \rightarrow 1 ; \theta \rightarrow \pi/2 \end{array} \right.$$

$$\therefore \beta(m, n) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow \beta(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} \cdot (\cos \theta)^{2n-1} d\theta$$

Proved

Gamma function:-

for $n > 0$, Gamma function is denoted as Γn and defined as.

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

NOTE :-

$$i) \Gamma n = \int_0^{\infty} e^{-y} y^{n-1} dy$$

$$ii) \Gamma(m+1) = \int_0^{\infty} e^{-x} x^{m+1-1} dx$$

$$iii) \Gamma(m+n) = \int_0^{\infty} e^{-x} x^{m+n-1} dx$$

Example :-

$$\boxed{\sqrt{\pi} = \Gamma \frac{1}{2}} = \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1} dx = \int_0^{\infty} \frac{dx}{e^x \sqrt{x}} //$$

Properties of Gamma function:-

Theorem-01 : $\Gamma n = \int_0^{\infty} e^{-zx} x^{n-1} z^n dx$

where $z \in \text{const.}$

or $\frac{\Gamma n}{z^n} = \int_0^{\infty} e^{-zx} x^{n-1} dx$

Proof: By Defⁿ. of Gamma function

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Put $x = zy$ [$z \in \text{const.}$]

$$\Rightarrow dx = z dy \quad \left| \begin{array}{l} \text{when} \\ x \rightarrow 0 ; y \rightarrow 0 \\ x \rightarrow \infty ; y \rightarrow \infty \end{array} \right.$$

$$\therefore \Gamma n = \int_0^{\infty} e^{-zy} (yz)^{n-1} z dy$$

$$= \int_0^{\infty} e^{-zy} y^{n-1} z^n dy \Rightarrow \underline{\Gamma n = \int_0^{\infty} e^{-zx} x^{n-1} z^n dx}$$

$$\Rightarrow \frac{\Gamma n}{z^n} = \int_0^{\infty} e^{-zy} y^{n-1} dy$$

$$\underline{\frac{\Gamma n}{z^n} = \int_0^{\infty} e^{-zx} x^{n-1} dx}$$

Proved

Theorem-02 : $\Gamma n = \int_0^1 \left(\log \frac{1}{x} \right)^{n-1} dx$

Proof: By Defⁿ of Gamma function

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\text{Put } x = \log \frac{1}{y} = -\log y \Rightarrow y = e^{-x} \quad \left| \begin{array}{l} \text{when} \\ x \rightarrow 0 ; y \rightarrow 1 \\ x \rightarrow \infty ; y \rightarrow 0 \end{array} \right.$$
$$\Rightarrow dx = -\frac{1}{y} dy = -\frac{1}{e^{-x}} dy$$

$$\therefore \Gamma n = \int_1^{\infty} \left(\log \frac{1}{y}\right)^{n-1} (-dy)$$

$$\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \quad // \text{ Proved}$$

$$\Rightarrow \Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dy //$$

≠ Prove $\Gamma 1 = 1$

Soln. We know that

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Rightarrow \Gamma 1 = \int_0^{\infty} e^{-x} x^{1-1} dx$$

$$= [e^{-x}]_0^{\infty} = -[e^{-\infty} - e^{-0}] = 1$$

$$\Rightarrow \Gamma 1 = 1 \quad \text{proved}$$

Some Formulae:-

$$i) \Gamma(n+1) = n \Gamma n = \Gamma n \quad (n \in \mathbb{R})$$

$$= n(n-1) \Gamma(n-1)$$

$$= n(n-1)(n-2) \Gamma(n-2)$$

$$\Gamma(n+1) = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 = \Gamma n$$

Example:-

$$1. \Gamma 4 = 4 \times 3 \times 2 \times 1 = \Gamma 5$$

$$2. \Gamma 3 = \Gamma(2+1) = 2 \Gamma 2 = 2 \times 1 = \Gamma 2$$

$$3. \Gamma \frac{3}{2} = \Gamma \left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma \frac{1}{2} = \frac{1}{2} \sqrt{\pi}$$

$$4. \Gamma \frac{5}{2} = \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{3}{4} \sqrt{\pi}$$

Relation between Beta and Gamma function:-

$$i) \boxed{\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}} ; m, n > 0$$

Proof: We have

$$\Gamma(n) = \int_0^{\infty} e^{-zx} x^{n-1} z^n dx$$

multiply by " $\Gamma(m) = \int_0^{\infty} e^{-z} z^{m-1} dz$ " both side

$$\Gamma(m) \Gamma(n) = \int_0^{\infty} e^{-zx} x^{n-1} z^n dx \times \int_0^{\infty} e^{-z} z^{m-1} dz$$

$$\Gamma(m) \Gamma(n) = \int_0^{\infty} \int_0^{\infty} e^{-z(x+1)} z^{m+n-1} x^{n-1} dx dz$$

$$\text{Put } z(x+1) = y$$

$$z = \frac{y}{1+x}$$

$$\begin{aligned} \text{When } x \rightarrow 0 ; y &\rightarrow 0 \\ x \rightarrow \infty ; y &\rightarrow \infty \end{aligned}$$

$$\Rightarrow dz = \frac{1}{x+1} dy$$

$$\Gamma(m) \Gamma(n) = \int_0^{\infty} \int_0^{\infty} e^{-y} \left(\frac{y}{x+1} \right)^{m+n-1} \times x^{n-1} \frac{1}{x+1} dx dy$$

$$= \int_0^{\infty} e^{-y} y^{m+n-1} dy \times \int_0^{\infty} \frac{x^{n-1}}{(x+1)^{m+n}} dx$$

$$\Gamma(m) \Gamma(n) = \Gamma(m+n) \cdot \beta(m, n)$$

$$\Rightarrow \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{Proved} //$$

Double Integral Concept

$$\text{i)} \quad \int_a^b \int_c^d f(x,y) dx dy = \int_a^b \left\{ \int_c^d f(x,y) dx \right\} dy \\ = \int_a^b \left\{ \int_c^d f(x,y) dy \right\} dx$$

$$\text{ii)} \quad \int_a^b \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx dy = \int_a^b \left\{ \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dy \right\} dx$$

$$\text{iii)} \quad \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) dx dy = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x,y) dx \right\} dy$$

Note:- When integrate w.r.t x keep y as const. and vice-versa.

Q Prove that $\Gamma_{\frac{1}{2}} = \sqrt{\pi}$

Soln:- We have

$$\beta(m,n) = 2 \int_0^{\pi/2} \sin^{2(m-1)} \theta \cdot \cos^{2(n-1)} \theta d\theta$$

$$\frac{\Gamma_m \Gamma_n}{\Gamma_{m+n}} = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\text{Put } m=n=\frac{1}{2}$$

$$\therefore \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma_{\frac{1}{2}} \Gamma_{\frac{1}{2}}}{\Gamma_{\frac{1}{2}+\frac{1}{2}}} = 2 \int_0^{\pi/2} \sin^0(\theta) \cos^0(\theta) d\theta$$

$$\Rightarrow \frac{(\Gamma_{\frac{1}{2}})^2}{\Gamma_1} = 2 [0]_0^{\pi/2}$$

$$\Rightarrow \Gamma_{\frac{1}{2}} = \sqrt{2(\pi/2 - 0)}$$

$$\Rightarrow \Gamma_{\frac{1}{2}} = \sqrt{\pi} \quad \text{proved}$$

Legendre's duplication formula :-

$$\boxed{\Gamma(m) \Gamma(m + \frac{1}{2}) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1}}} \quad m > 0$$

Proof:- We know that

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2n-1} \theta \, d\theta \quad ; m > 0 \quad \dots \dots \textcircled{I}$$

Case 1

$$\text{Let } 2n-1 = 0 \Rightarrow n = \frac{1}{2}$$

$$2 \int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^0 \theta \, d\theta = \frac{\Gamma(m) \Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \quad \dots \dots \textcircled{II}$$

Case-02

$$\text{Let } m=n=m$$

$$2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2m-1} \theta \, d\theta = \frac{\Gamma(m) \Gamma(m)}{\Gamma(2m)}$$

$$\Rightarrow 2 \int_0^{\pi/2} (\sin \theta \cos \theta)^{2m-1} d\theta = \frac{(\Gamma(m))^2}{\Gamma(2m)}$$

$$\Rightarrow \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta = \frac{\Gamma(m)^2}{\Gamma(2m)}$$

$$\Rightarrow 2 \int_0^{\pi/2} \sin^{2m-1} \theta \, d\theta = \frac{\Gamma(m)^2 2^{2m-1}}{\Gamma(2m)} \quad \left[\int_0^{\pi/2} \sin 2\theta \, d\theta = \int_0^{\pi/2} \sin \theta \, d\theta \right] \quad \dots \dots \textcircled{III}$$

$$\frac{\Gamma(m) \sqrt{\pi}}{\Gamma(m + \frac{1}{2})} = \frac{\Gamma(m)^2 2^{2m-1}}{\Gamma(2m)} \Rightarrow \Gamma(m) \Gamma(m + \frac{1}{2}) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1}}$$

Proved

Q Express $\int_0^1 x^m (1-x^n)^p dx$ in term of Beta and Gamma function. And hence evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

Soln Let $I = \int_0^1 x^m (1-x^n)^p dx$ ----- (i)

Put $x^n = y \Rightarrow x = y^{\frac{1}{n}}$ when
 $x \rightarrow 0, y \rightarrow 0$
 $x \rightarrow 1, y \rightarrow 1$
 $dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

$$I = \int_0^1 (y^{\frac{1}{n}})^m (1-y)^p \frac{1}{n} y^{\frac{1}{n}-1} dy$$

$$= \frac{1}{n} \int_0^1 y^{\frac{m+1}{n}-1} (1-y)^{p+1-1} dy$$

$$I = \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right)$$

$$I = \frac{1}{n} \left[\frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + (p+1)\right)} \right] \text{ ----- (ii)}$$

Now

To evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$

Compare with eq (i)

$m=5, n=3, p=10$. Put in eq (ii)

$$I = \int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{3} \frac{\Gamma 12 \Gamma 11}{\Gamma 13} = \frac{1}{3} \frac{11! \times 11!}{13!}$$

$$= \frac{1}{396} //$$

Q Prove that $\int_{-\infty}^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{a}$; $a > 0$

Formula :- $\left[\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ --- if } f(x) \text{ is even} \right.$
 $\left. = 0 \text{ --- if } f(x) \text{ is odd} \right]$

$$\Rightarrow 2 \int_0^{\infty} e^{-a^2 x^2} dx$$

Put $a^2 x^2 = y$

$$x \rightarrow 0; y \rightarrow 0$$

$$x \rightarrow \infty; y \rightarrow \infty$$

$$x^2 = \frac{1}{a^2} y \Rightarrow x = \frac{1}{a} y^{1/2}$$

$$\Rightarrow dx = \frac{1}{2a} y^{-1/2} dy$$

$$\Rightarrow I = \int_{-\infty}^{\infty} e^{-a^2 x^2} dx = 2 \int_0^{\infty} e^{-y} \frac{1}{2a} y^{-1/2} dy$$

$$= \frac{1}{a} \int_0^{\infty} e^{-y} y^{1/2-1} dy$$

$$= \frac{1}{a} \Gamma_{1/2}$$

$$I = \frac{\sqrt{\pi}}{a} \text{ proved, } a > 0$$

Q Prove that $\int_0^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a}$; $a > 0$

Soln.

put $a^2 x^2 = y \Rightarrow x = \frac{y^{1/2}}{a}$

$$\Rightarrow dx = \frac{1}{2a} y^{-1/2} dy$$

$$\Rightarrow I = \int_0^{\infty} e^{-y} \frac{1}{2a} y^{-1/2} dy$$

$$= \frac{1}{2a} \int_0^{\infty} e^{-y} y^{1/2-1} dy$$

$$I = \frac{\Gamma_{1/2}}{2a} = \frac{\sqrt{\pi}}{2a} \text{ proved, } a > 0$$

Q Prove that

i) $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$

Soln: Proof: LHS = $\beta(m, n) = \frac{m \cdot n}{m+n}$ --- (1)

RHS = $\beta(m+1, n) + \beta(m, n+1)$

RHS = $\frac{(m+1) \cdot n}{m+n+1} + \frac{m \cdot (n+1)}{m+n+1}$

= $\frac{(m+1) \cdot n + m \cdot (n+1)}{(m+n+1)}$

= $\frac{m \cdot n + n + m \cdot n + m}{(m+n+1)}$

= $\frac{(m+n) \cdot (m+n)}{(m+n+1)}$

RHS = $\frac{m \cdot n}{m+n} = \text{LHS}$ Proved //

ii) $\beta \frac{(m+1, n)}{m} = \beta \frac{(m, n+1)}{n} = \beta \frac{(m, n)}{m+n}$

Proof:- Let $T_1 = \beta(m+1, n)/m$

= $\frac{(m+1) \cdot n}{m(m+n+1)} = \frac{\cancel{m} \cdot n}{\cancel{m}(m+n+1)} = \frac{\beta(m, n)}{(m+n)}$ --- (1)

$T_2 = \beta(m, n+1)/n$

= $\frac{m \cdot (n+1)}{n(m+n+1)} = \frac{m \cdot \cancel{n}}{(m+n) \cdot \cancel{n}} = \frac{\beta(m, n)}{(m+n)}$ --- (2)

from (1) and (2)

$\beta(m+1, n)/m = \beta(m, n+1)/n = \beta(m, n)/(m+n)$

Proved //

$$iii) \frac{\beta(m, n+1)}{\beta(m, n)} = \frac{n}{m+n}$$

$$\text{Proof: LHS} = \frac{\beta(m, n+1)}{\beta(m, n)} = \frac{\frac{\Gamma(m) \Gamma(n+1)}{\Gamma(m+n+1)}}{\frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}} = \frac{n \Gamma(m) \Gamma(n)}{m \Gamma(m+n)} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\text{LHS} = \frac{n}{m+n} = \text{RHS} \quad \text{proved} //$$

Q Prove that $\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ $0 < n < 1$

$$\text{where } \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$$

Soln: Proof: we have

$$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \dots \text{--- (1)}$$

$$\text{let } m+n=1$$

$$m = n-1 \quad \text{Put in (1)}$$

$$\frac{\Gamma(1-n) \Gamma(n)}{\Gamma(1)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^1} dx$$

$$\Rightarrow \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{Proved} //$$

Q Prove that

$$\int_0^{\pi/2} \tan^n x dx = \frac{\pi}{2} * \sec \frac{n\pi}{2}$$

$$\text{Soln: let } I = \int_0^{\pi/2} \tan^n x dx$$

$$= \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x} dx$$

$$I = \int_0^{\pi/2} \sin^n x \cos^n x dx \quad \dots \text{--- (1)}$$

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{1}{2} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{--- (1)}$$

Comparing LHS from (1) & (11)

$$2m-1 = n$$

$$2N-1 = -n$$

$$\Rightarrow m = \frac{n+1}{2}$$

$$\Rightarrow N = \frac{1-n}{2}$$

Put in (11)

$$\text{let } I = \int_0^{\pi/2} \sin^n \theta \cos^{-n} \theta d\theta = \frac{\frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1-n}{2}\right)}{\Gamma\left(\frac{n+1}{2} + \frac{1-n}{2}\right)}$$

$$\Rightarrow I = \frac{\frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right)}{\Gamma(1)}$$

$$= \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \frac{\pi}{\sin\left(\frac{n+1}{2}\pi\right)}$$

$$= \frac{1}{2} \frac{\pi}{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)} = \frac{\pi}{2 \cos \frac{n\pi}{2}} = \frac{\pi}{2} \sec \frac{n\pi}{2}$$

Hence,

$$\int_0^{\pi/2} \sin^n \theta \cos^{-n} \theta d\theta = \frac{\pi}{2} \sec \frac{n\pi}{2} \quad \text{Proved} //$$

Q Evaluate : $\sqrt{-\frac{1}{2}}, \sqrt{\frac{3}{2}}, \sqrt{-\frac{5}{2}}, \dots$

Solⁿ: We know that

$$\overline{m+1} = n \overline{m}$$

$$\overline{m} = \frac{1}{n} \overline{m+1} \quad \text{--- (1)}$$

i) Put $n = -\frac{1}{2}$ in (1)

$$\Gamma_{-\frac{1}{2}} = \frac{1}{(-\frac{1}{2})} \Gamma_{-\frac{1}{2}+1} = -2 \Gamma_{\frac{1}{2}} = -2\sqrt{\pi}$$

ii) Put $n = -\frac{3}{2}$ in (1)

$$\Gamma_{-\frac{3}{2}} = \frac{1}{-\frac{3}{2}} \Gamma_{-\frac{3}{2}+1} = -\frac{2}{3} \Gamma_{-\frac{1}{2}} = -\frac{2}{3}(-2\sqrt{\pi}) = \frac{4}{3}\sqrt{\pi}$$

iii) Put $n = -\frac{5}{2}$ in (1)

$$\Gamma_{-\frac{5}{2}} = \frac{1}{-\frac{5}{2}} \Gamma_{-\frac{5}{2}+1} = -\frac{2}{5} \Gamma_{-\frac{3}{2}} = -\frac{2}{5} \left(\frac{4}{3}\sqrt{\pi} \right) = -\frac{8}{15}\sqrt{\pi}$$

Q Evaluate: $\int_0^{\infty} \frac{x^8(1+x^6)}{(1+x)^{24}} dx$

Soln.

let $I = \int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx + \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}} dx$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} dx + \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

$$\left[\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m, n) \right]$$

$$I = \beta(9, 15) + \beta(15, 9)$$

$$= 2\beta(9, 15)$$

$$= 2 \frac{\Gamma(9) \Gamma(15)}{\Gamma(9+15)}$$

$$= \frac{2 \cdot 18 \cdot 114}{129} = \frac{2 \times 8 \times 7 \times 8 \times 8 \times 4 \times 3 \times 2}{23 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15} = \frac{1}{36738}$$

Q Evaluate $\int_0^{\infty} \frac{1}{1+x^4} dx$

Soln. Let $I = \int_0^{\infty} \frac{1}{1+x^4} dx$

Put $x^4 = t$ $x \rightarrow 0; t \rightarrow 0$
 $\Rightarrow x = t^{1/4}$ $x \rightarrow \infty; t \rightarrow \infty$

$\Rightarrow dx = \frac{1}{4} t^{-3/2} dt$

$I = \int_0^{\infty} \frac{1}{1+t} \cdot \frac{1}{4} t^{-3/2} dt$

$= \frac{1}{4} \int_0^{\infty} \frac{t^{\frac{1}{4}-1}}{(1+t)^{\frac{1}{4}+\frac{3}{4}}} dt = \frac{1}{4} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$I = \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{4} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1-\frac{1}{4}\right) = \frac{1}{4} \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} //$

Q Prove that $\int_0^1 x^2 (1-x)^3 dx = \frac{1}{60}$

Soln. Let $I = \int_0^1 x^2 (1-x)^3 dx$

$= \int_0^1 x^{3-1} (1-x)^{4-1} dx$

$= \beta(3, 4) = \frac{\Gamma(3) \Gamma(4)}{\Gamma(7)} = \frac{2! \times 3!}{6!} = \frac{1}{60}$

Proved //

Q Prove that $\int_0^{\infty} e^{-ax} x^{5/2} dx = \frac{15}{1024}$

Soln Proof: Let $I = \int_0^{\infty} e^{-ax} x^{5/2} dx$

Put $4x = y$ $x \rightarrow 0; y \rightarrow 0$
 $\Rightarrow dx = \frac{1}{4} dy$ $x \rightarrow \infty; y \rightarrow \infty$

$\therefore I = \int_0^{\infty} e^{-y} \left(\frac{y}{4}\right)^{5/2} \cdot \frac{1}{4} dy = \frac{1}{4^{7/2}} \int_0^{\infty} e^{-y} y^{7/2} dy$

$$\Rightarrow I = \frac{1}{128} \sqrt{\frac{5}{2}} = \frac{\sqrt{5+1}}{128}$$

$$= \frac{\frac{5}{2} \times \frac{3 \times \sqrt{1}}{2}}{128} = \frac{1}{2} \frac{15\pi}{512} = \frac{15\pi}{1024} //$$

Q Evaluate $\int_0^{\infty} e^{-\sqrt{3}x} \sqrt{x} dx$

Soln:- let $I = \int_0^{\infty} e^{-\sqrt{3}x} \sqrt{x} dx$

Put $\sqrt{3}x = t$
 $\Rightarrow x = \frac{t^2}{\sqrt{3}^2}$

$x \rightarrow 0 ; t \rightarrow 0$
 $x \rightarrow \infty ; t \rightarrow \infty$

$\Rightarrow dx = \frac{2t}{\sqrt{3}^2} dt$

$$\Rightarrow I = \int_0^{\infty} e^{-t} \frac{t}{\sqrt{3}} \frac{2t}{3} dt$$

$$= \frac{2}{3\sqrt{3}} \int_0^{\infty} e^{-t} t^{3-1} dt$$

$$\Rightarrow I = \frac{2}{3\sqrt{3}} \Gamma(3) = \frac{2 \times 2}{3\sqrt{3}} = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9} //$$

Q $\int_0^2 x(8-x^3)^{1/3} dx = \frac{16\pi}{9\sqrt{3}}$ Prove?

Soln. let $I = \int_0^2 x(8-x^3)^{1/3} dx$

Put $x^3 = 8y$

$x = (8y)^{1/3}$

$dx = 8^{1/3} \frac{1}{3} y^{1/3-1} dy$

If $x \rightarrow 0$ $y \rightarrow 0$

$y \rightarrow 2$ $y \rightarrow 1$

$$I = \int_0^1 (8y)^{1/3} (8-8y)^{1/3} \frac{2}{3} y^{1/3-1} dy = \frac{8}{3} \int_0^1 y^{2/3-1} (1-y)^{1/3-1} dy$$

$$= \frac{8}{3} B\left(\frac{2}{3}, \frac{1}{3}\right) //$$

$$\Rightarrow I = \frac{8}{3} \beta\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$= \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3} + \frac{4}{3}\right)} = \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{6}{3}\right)} = \frac{8}{3} \frac{\Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{2}{3}\right)}{\Gamma(2)}$$

$$I = \frac{8}{3} \left\{ \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{2}{3}\right) \right\} = \frac{8}{4} \Gamma\left(\frac{1}{3}\right) \Gamma\left(1 - \frac{1}{3}\right)$$

$$I = \frac{8}{9} \frac{\pi}{\sin \frac{4}{3}\pi} = \frac{8}{9} \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{16\pi}{9\sqrt{3}}$$

Proved //

Double integral:-

Rule 1 (i) $\int_a^b \int_c^d f(x, y) dx dy = \int_a^b \left[\int_c^d f(x, y) dx \right] dy$

(ii) $\int_a^b \int_{g_2(y)}^{g_1(y)} f(x, y) dx dy = \int_a^b \left\{ \int_{g_2(y)}^{g_1(y)} f(x, y) dx \right\} dy$

(iii) $\int_a^b \left\{ \int_{f_2(x)}^{f_1(x)} f(x, y) dy \right\} dx = \int_a^b \left[\int_{f_2(x)}^{f_1(x)} f(x, y) dy \right] dx$

Note:- i) If we integrate w.r to x keeping y as constant.
 (ii) 'if we -1 -' to y keeping x as -1 -'.

Evaluate:-

Ex.

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx dy}{1+x^2+y^2}$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{y^2 + (1+x^2)} dy \right] dx$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{y^2 + (\sqrt{1+x^2})^2} dy \Bigg] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x^2}} \tan^{-1} 0 \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \tan \frac{\pi}{4} - \frac{1}{\sqrt{1+x^2}} \tan^{-1} \tan 0 \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \left[\log(x + \sqrt{1+x^2}) \right]_0^1$$

$$= \frac{\pi}{4} [\log (1+\sqrt{2}) - \log 1]$$

$$I = \frac{\pi}{4} \log (1+\sqrt{2})$$

Q.11 $\int_0^1 \left\{ \int_0^{x^3} e^{y/x} dy \right\} dx$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^3} dx$$

$$= \int_0^1 x \left[e^{y/x} \right]_0^{x^3} dx$$

$$= \int_0^1 x \left(e^{x^2} - e^0 \right) dx$$

$$= \int_0^1 x (e^{x^2} - 1) dx$$

$$= \int_0^1 x e^{x^2} dx - \int_0^1 x dx$$

$$I = \left[x e^{x^2} - 1 \cdot e^{x^2} \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$

$$= \left\{ (e - e) - (0 - e^0) \right\} - \left\{ \frac{1}{2} - 0 \right\}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{I = \frac{1}{2}}$$

$$③ \int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-y^2}} dx dy$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} y \right]_0^1 dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} 1 - \sin^{-1} 0 \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\frac{\pi}{2} - 0 \right] dx$$

$$= \frac{\pi}{2} \left[\sin^{-1} x \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1} 1 - \sin^{-1} 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{2} - 0 \right]$$

$$\boxed{I = \frac{\pi^2}{4}}$$

(i) eqⁿ for x axis $y = 0$

eqⁿ for y axis $x = 0$

(ii) eqⁿ || to x axis $y = \text{Constant}$

(iii) eqⁿ || to y axis $x = \text{Constant}$

Maths

26" Evaluate $\iint_R x^2 y^2 dx dy$; where R is the region

bounded by $x=0, y=0$ and $x^2+y^2=a^2$

Soln Let $I = \iint_R x^2 y^2 dx dy$

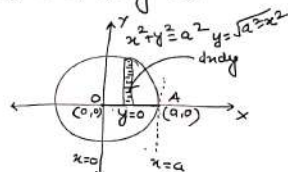
$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y^2 dx dy$$

$$= \int_0^a x^2 \left\{ \frac{y^3}{3} \right\}_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{3} \int_0^a x^2 \left((a^2-x^2)^{3/2} - 0 \right) dx$$

$$I = \frac{1}{3} \int_0^a x^2 (a^2-x^2)^{3/2} dx$$

Let $x = a \sin \theta$
 $dx = a \cos \theta d\theta$



x varies from 0 to a .
 y " " " 0 to $\sqrt{a^2-x^2}$

$$I = \frac{1}{3} \int_0^{\pi/2} a^2 \sin^2 \theta (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos \theta d\theta$$

$$= \frac{a^6}{3} \int_0^{\pi/2} \sin^2 \theta (\cos^2 \theta)^{3/2} \cos \theta d\theta$$

$$= \frac{a^6}{3} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta \quad \text{--- (1)}$$

NOTE:

(i) $\int dx dy$ Vertical Strip

We take limit of x in constant and limit of y in variable

(ii) $\int dx dy$ Horizontal Strip

We take limit of y in constant and limit of x in variable

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\frac{1}{2} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \quad \text{--- (ii)}$$

$$\therefore 2m-1 = 2 \quad m = \frac{3}{2}$$

$$2n-1 = 4 \quad n = \frac{5}{2}$$

$$\frac{1}{2} \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2} + \frac{5}{2})} = \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$$

$$\frac{1}{2} \frac{\frac{1}{2} \sqrt{\pi} \cdot \frac{3}{2} \frac{1}{2} \sqrt{\pi}}{\Gamma(4)} = \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$$

$$\frac{2 \times \pi}{16 \times 6} = \frac{\pi}{32} = \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$$

from (1) & (3)

$$I = \frac{a^6}{3} \cdot \frac{\pi}{32}$$

$$= \frac{\pi}{96} a^6 \text{ Ans}$$

from (2)

$$\frac{1}{2} \frac{\sqrt{\frac{1}{2}} \sqrt{\frac{3}{2}}}{\sqrt{\frac{1}{2} + \frac{3}{2}}} = \int_0^{\pi/2} \sin^2 \theta \cdot \cos^2 \theta d\theta$$

$$x + \frac{b}{a} = \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx$$

$$\frac{1}{2} \frac{\sqrt{\pi} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{\sqrt{2}} = \text{--- do ---}$$

let $x = a \sin \theta$
 $dx = a \cos \theta d\theta$

$$\frac{\sqrt{\pi} \cdot \sqrt{\pi}}{4 \sqrt{1}} = \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

If $x \rightarrow 0$ $\theta \rightarrow 0$
 $x \rightarrow a$ $\theta \rightarrow \pi/2$

$$\frac{\pi}{4} = \int_0^{\pi/2} \sin^2 \theta \cdot \cos^2 \theta d\theta$$

$$I = \frac{b}{a} \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta$$

from (1) & (3)

$$I = ab \cdot \frac{\pi}{4}$$

Ans

$$\begin{aligned} I &= \frac{b}{a} a^2 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= ab \int_0^{\pi/2} \sin^2 \theta \cdot \cos^2 \theta d\theta \end{aligned}$$

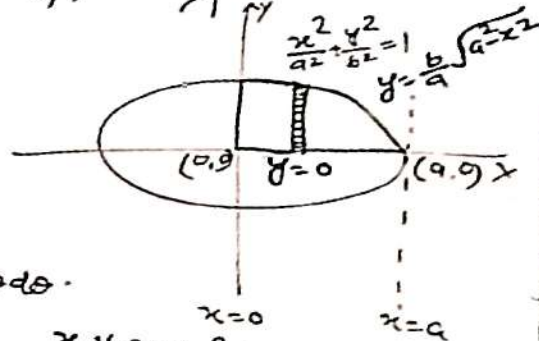
$$\left(\frac{1}{2} B(m, n) = \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \right)$$

$$\frac{1}{2} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$\therefore 2m-1=0, 2n-1=2$
 $m=1/2, n=3/2$

$$\int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

Evaluate $\iint_R dx dy$ where
 R is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 of positive quadrant.



x varies from 0 to a .
 y varies from 0 to $\frac{b}{a} \sqrt{a^2 - x^2}$.

$$I = \iint_R 1 \cdot dx dy = \int_0^a \left\{ \int_0^{\frac{b}{a} \sqrt{a^2 - x^2}} 1 \cdot dy \right\} dx$$

$$= \int_0^a \left\{ y \right\}_0^{\frac{b}{a} \sqrt{a^2 - x^2}} dx$$

Remark

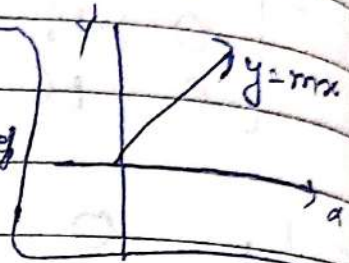
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i) Eqⁿ of straight line passing through the origin

$$y = mx$$

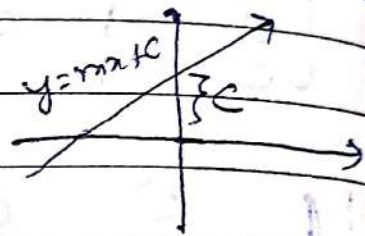
↳ slope



ii) Eqⁿ of st. line which cuts the intercept on y axis. i.e.

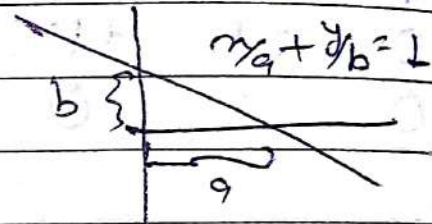
$$y = mx + c \rightarrow \text{intercept}$$

↳ slope



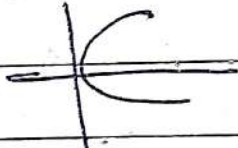
iii) Eqⁿ of st line which cuts the intercept on both axes.

$$\text{P.S. } \frac{x}{a} + \frac{y}{b} = 1$$

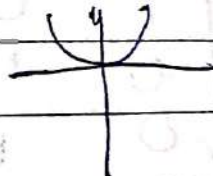


1. Parabola.

a) $y^2 = 4ax$



c) $x^2 = 4y$



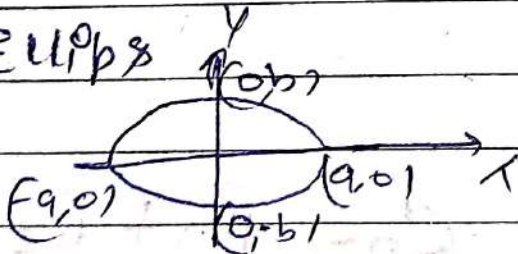
b) $y^2 = -4ax$



d) $x^2 = -4y$



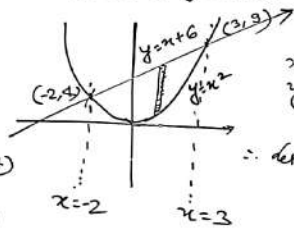
2. Ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Q3: Evaluate $\iint_R x \cdot dy dx$ where R is the region bounded by $y = x^2$ and st. line $y = x+6$.

$y = x^2, y = x+6$
 $x^2 = x+6$
 $x^2 - x - 6 = 0$
 $x = -2, 3$
 $\square x = -2, y = 4 \rightarrow (-2, 4)$
 $\square x = 3, y = 9 \rightarrow (3, 9)$

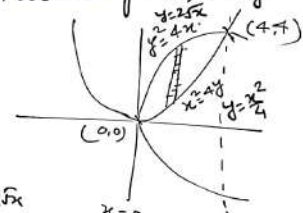


x varies from -2 to 3
 y " from x^2 to $x+6$
 $\therefore \text{Ans} = \int_{-2}^3 \int_{x^2}^{x+6} x \cdot dy dx$
 $= \int_{-2}^3 [x \cdot y]_{x^2}^{x+6} dx$
 $= \int_{-2}^3 [x(x+6) - x \cdot x^2] dx$
 $= \int_{-2}^3 [x^2 + 6x - x^3] dx$
 $= [\frac{x^3}{3} + 3x^2 - \frac{x^4}{4}]_{-2}^3$
 $= \frac{125}{12}$

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$
 $y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$

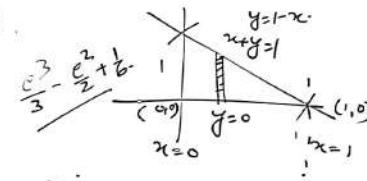
Q7: $\iint_R y \cdot dy dx$; over the region bounded by the parabola $y^2 = 4x, x^2 = 4y$.

$y^2 = 4x$
 $x^2 = 4y$
 $(0,0) (4,4)$

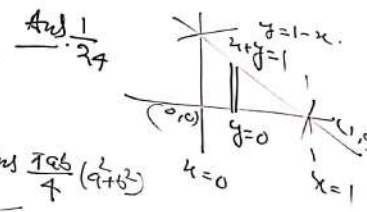


$\therefore \text{Ans} = \int_0^4 \int_{x^2/4}^{\sqrt{4x}} y \cdot dy dx$
 $= \frac{48}{5}$

Q5: $\iint_R e^{2x+3y} \cdot dy dx$; over the region bounded by the triangle $x=0, y=0$ and $x+y=1$



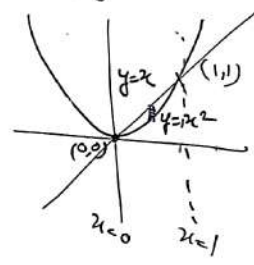
Q6: $\iint_R xy \cdot dy dx$; bounded by $x+y \leq 1$ in first quadrant.



Ans $\frac{1}{24}$

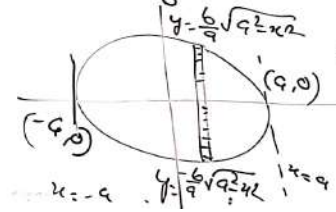
Q4: $\iint_R xy(x+y) \cdot dy dx$ over the region bounded by the curve $y=x^2$ and $y=x$.

$y=x, y=x^2$
 $x^2=x$
 $x(x-1)=0$
 $x=0, x=1$
 $\square x=0, y=0 \rightarrow (0,0)$
 $\square x=1, y=1 \rightarrow (1,1)$



$\therefore \text{Ans} = \int_0^1 \int_{x^2}^x (x^2y + xy^2) dy dx$
 $= \frac{3}{5}$

Q8: Evaluate $\iint_R (x+y)^2 \cdot dy dx$ over the region bounded by ellipse $\frac{x^2}{9} + \frac{y^2}{6} = 1$



Ans $\frac{796}{4} (9+6^2)$

Q3 Evaluate $\iint_R x \, dy \, dx$ where R is the region bounded by $y = x^2$ and st. line $y = x + 6$

Soln: $y = x^2, y = x + 6$

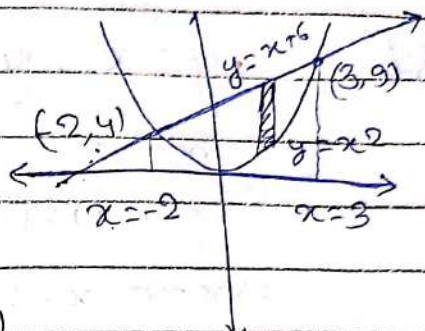
$$x^2 = x + 6$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x = -2, 3$$

At $x = -2, y = 4$ $(x, y) = (-2, 4)$

At $x = 3, y = 9$ $(x, y) = (3, 9)$



Now

$$\iint_R x \, dy \, dx = \int_{-2}^3 \int_{x^2}^{x+6} x \, dx \, dy$$

$$= \int_{-2}^3 \left[\int_{x^2}^{x+6} x \, dy \right] dx$$

$$= \int_{-2}^3 [x \cdot y]_{y=x^2}^{y=x+6} dx$$

$$= \int_{-2}^3 x(x+6-x^2) dx$$

$$= \left[\frac{x^3}{3} + \frac{6x^2}{2} - \frac{x^4}{4} \right]_{-2}^3$$

$$= \left(9 + 27 - \frac{81}{4} \right) - \left(-\frac{8}{3} + 12 - 4 \right)$$

$$= \left(\frac{36 + 108 - 81}{4} \right) - \left(\frac{36 - 8 - 12}{3} \right) = \frac{125}{12}$$

40 $\iint_R xy(x+y) dy dx$ over the region bounded by the curve $y=x^2$ and $y=x$

Solⁿ: $y=x, y=x^2$

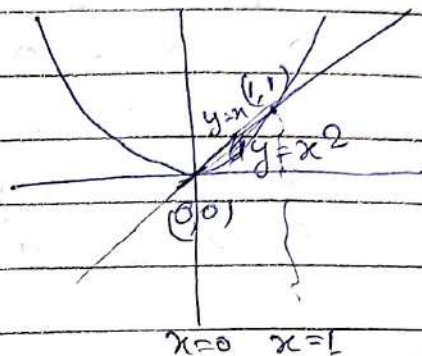
$$x^2=x$$

$$x(x-1)=0$$

$$\Rightarrow x=0, x=1$$

if $x=0, y=0$ $(x,y) \equiv (0,0)$

if $x=1, y=1$ $(x,y) \equiv (1,1)$



$$\therefore I = \int_0^1 \int_{x^2}^x (x^2y + xy^2) dy dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2} + x \frac{y^3}{3} \right]_{x^2}^x dx$$

$$= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx$$

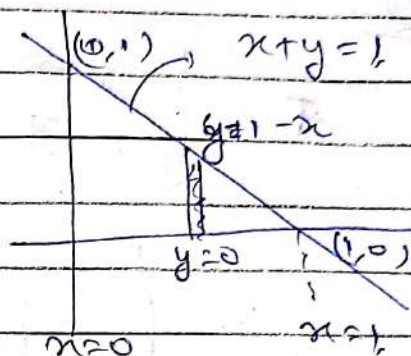
$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} = \frac{84+56-60-35}{840} = \frac{45}{840} = \frac{9}{168}$$

5. $\iint_R e^{2x+3y} dx dy$; over the region

bounded by the triangle $x=0, y=0, x+y=1$

$$I = \int_0^1 \int_{y=0}^{y=1-x} (e^{2x} \cdot e^{3y}) dx dy$$



$$= \int_0^1 e^{2x} \left[\int_0^{1-x} e^{3y} dy \right] dx$$

$$= \int_0^1 e^{2x} \left[\frac{e^{3y}}{3} \right]_0^{1-x} dx$$

$$= \frac{1}{3} \int_0^1 e^{2x} [e^{3(1-x)} - e^0] dx$$

$$= \frac{1}{3} \int_0^1 [e^3 (e^{-x}) - e^{2x}] dx$$

$$= \frac{1}{3} \left[-e^3 e^{-x} - \frac{1}{2} e^{2x} \right]_0^1$$

$$= \frac{1}{3} \left[(-e^3 e^{-1} - \frac{e^2}{2}) - (-e^3 - \frac{1}{2}) \right]$$

$$= \frac{1}{3} \left[-e^2 - \frac{1}{2} e^2 + e^3 + \frac{1}{2} \right] = \frac{e^3}{3} - \frac{e^2}{2} + \frac{1}{6} //$$

6Q $\iint_R xy \, dx \, dy$, bounded by $x+y \leq 1$

In 1st quadrant.

Soln

$$x+y=1$$

$$I = \int_0^1 \left[\int_{y=0}^{y=1-x} xy \, dy \right] dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} dx$$

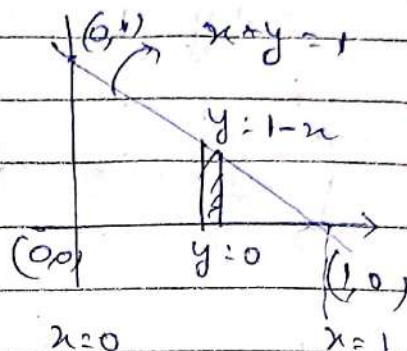
$$= \int_0^1 x \left[\frac{(1-x)^2}{2} - 0 \right] dx$$

$$= \frac{1}{2} \int_0^1 (x^3 + 2x^2 + x) dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - 2 \cdot \frac{x^3}{3} + \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{4} - 2 \times \frac{1}{3} + \frac{1}{2} \right) - (0) \right]$$

$$= \frac{1}{2} \left[\frac{3-8+6}{12} \right] = \frac{1}{24} //$$



Q7) $\iint_R y \cdot dx \cdot dy$; over the region bounded by the parabola. $y^2 = 4x$, $x^2 = 4y$.

$$y^2 = 4x, x^2 = 4y$$

$$(0,0) (4,4)$$

$$I = \int_0^4 \int_{x/4}^{2\sqrt{x}} y \cdot dy \cdot dx$$

$$= \int_0^4 \left[\frac{y^2}{2} \right]_{x/4}^{2\sqrt{x}} dx$$

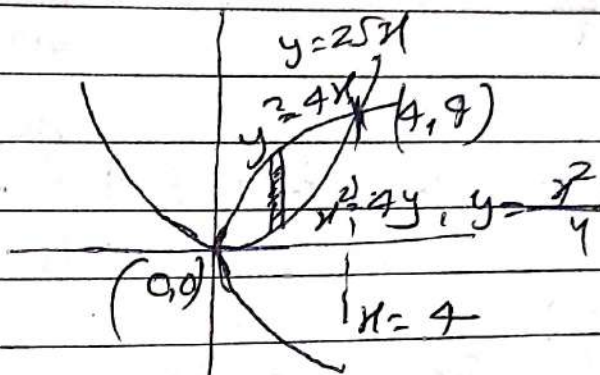
$$= \int_0^4 \left(\frac{(2\sqrt{x})^2}{2} - \frac{(x/4)^2}{2} \right) dx$$

$$= \int_0^4 \left(\frac{4x}{2} - \frac{x^2}{32} \right) dx$$

$$= \left[\frac{4x^2}{4} - \frac{x^3}{160} \right]_0^4$$

$$= \left[\left(16 - \frac{1024}{160} \right) - (0-0) \right]$$

$$= \frac{160-64}{10} = \frac{96}{10} = \frac{48}{5}$$



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Volume

Triple Integral

$$V = \iiint_V 1 \cdot dv$$

$$= \iiint_V 1 \cdot dx dy dz$$

Rule of Triple Integral

Rule I:

$$I = \int_a^b \int_c^d \int_e^f f(x, y, z) dx dy dz.$$

$$= \int_a^b \int_c^d \left\{ \int_e^f f(x, y, z) dx \right\} dy \int dz$$

$$\text{OR}$$

$$= \int_a^b \int_c^d \left\{ \int_e^f f(x, y, z) dz \right\} dy \int dx.$$

$$\text{OR}$$

$$= \int_a^b \int_c^d \left\{ \int_e^f f(x, y, z) dy \right\} dx \int dz \quad \text{--- etc.}$$

$$\text{Rule II: } \int_a^b \int_{f_2(x)}^{f_1(x)} \int_{\phi_2(x, y)}^{\phi_1(x, y)} f(x, y, z) dx dy dz$$

$$= \int_a^b \int_{f_2(x)}^{f_1(x)} \left\{ \int_{\phi_2(x, y)}^{\phi_1(x, y)} f(x, y, z) dz \right\} dy \int dx.$$

NOTE:- In Triple Integral any single variable so first we integrate w.r.to z.

- (ii) $\left. \begin{array}{l} (y, z) \text{ first w.r.to } x \\ (z, x) \text{ first w.r.to } y \end{array} \right\}$

11) IB We Integrate w.r to x keep in y, z a constant
to y " " z " "
to z " " x, y "

Q Evaluate

$$I = \int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \cdot dx dy dz$$

$$= \int_0^1 \int_0^1 \int_0^1 e^x (e^y (e^z dz) dy) dx$$

$$= \int_0^1 \int_0^1 e^x \cdot e^y \int_0^1 e^z dz dy dx$$

$$= \int_0^1 \int_0^1 e^x (e^y dy) (e^1 - e^0) dx$$

$$= (e-1) \int_0^1 e^x \cdot \left\{ e^y \int_0^1 dx \right\}$$

$$= (e-1)^2 \left\{ e^x \right\}_0^1$$

$$= (e-1)^3$$

Q1) To find the volume of solid bounded by $x=0, y=0, z=0$ and $x+y+z=1$

Limit are: $z=0$ $z=1-x-y$

$y=0$ $y=1-x$

$x=0$ $x=1$

$$V = \iiint_V dv = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \cdot dz dy dx$$

$$= [z]_0^{1-x-y} dy dx$$

$$= [1-x-y] dy dx$$

$$\int_0^1 \int_0^{1-x} [(1-x)-y] dy dx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 (1-x)^2 - \left(\frac{1-x}{2} \right)^2 dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 dx$$

$$= \frac{1}{2} [1+x^2-2x] dx$$

$$= \frac{1}{2} \left[x + \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^1 = \frac{1}{2} \left[1 + \frac{1}{3} - 1 \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{3} - 1 \right] = \frac{1}{6} \text{ Ans}$$

$$\text{Q. 11} \quad V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, dz \, dy \, dx$$

$$V = \int_0^1 \int_0^{1-x} \left[z \right]_0^{1-x-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[1-x-y \right] dy \, dx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2} \right) dx$$

$$= \int_0^1 \left[1 \cdot (1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[1(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[(1-x)(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2} \right] dx$$

Ex. 2

$$\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1 \cdot dx \, dy \, dz$$

$$= 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} 1 \cdot dz \right) dy \, dx$$

$$= 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[z \right]_0^{\sqrt{a^2-x^2-y^2}} dy \, dx$$

$$= 2 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\sqrt{a^2-x^2-y^2} \cdot dy \right] dx$$

$$= 2 \int_{-a}^a 2 \int_0^{\sqrt{a^2-x^2}} \left[\sqrt{(\sqrt{a^2-x^2})^2 - y^2} \cdot dy \right] dx$$

$$\therefore b = \sqrt{a^2-x^2}$$

$$= 4 \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{b^2-y^2} \, dy \, dx$$

$$= 4 \int_{-a}^a \left[\frac{y}{2} \sqrt{b^2 - y^2} + \frac{b^2}{2} \sin^{-1} \frac{y}{b} \right] dy$$

$$= 4 \int_{-a}^a \frac{\sqrt{a^2 - x^2}}{2} \left[\sqrt{(a^2 - x^2) - (a^2 - x^2)} + \frac{a^2 - x^2}{2} \sin^{-1} \frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} \right]$$

$$= 4 \int_{-a}^a \left[0 + \frac{a^2 - x^2}{2} \sin^{-1} \frac{x}{a} \right] dx$$

$$V = 4 \times \frac{\pi}{4} \int_{-a}^a (a^2 - x^2) dx$$

$$= 2\pi \int_0^a (a^2 - x^2) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right]$$

$$V = 2\pi \frac{2a^3}{3} = \frac{4\pi a^3}{3}$$

3. ϕ ~~reduces~~

$$V = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx dy dz$$

$$= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left\{ \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz \right\} dy dx$$

$$= \int_{-a}^a 2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2}} dy dx$$

$$= 2 \int_{-a}^a \left\{ \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dy \right\} dx$$

$$= 4 \int_{-a}^a \left\{ \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dy \right\} dx$$

$$= 4 \int_{-a}^a \left[\frac{y}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{-a}^a \left(\frac{a^2-x^2}{2} + \frac{a^2}{2} \sin^{-1} \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \right) dx$$

$$= 4 \left[2 \int_0^a \frac{a^2-x^2}{2} dx + a^2 \int_0^a \sin^{-1} \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} dx \right]$$

$$= 4 \left[\left\{ a^2 x - \frac{x^3}{3} \right\}_0^a - a^2 \int_0^a \sin^{-1} \sqrt{a^2 - x^2} dx \right]$$

Put $x = a \cos \theta$

$$\Rightarrow dx = -a \sin \theta d\theta \quad \left| \begin{array}{l} x \rightarrow 0 \\ \theta \rightarrow \pi/2 \\ x \rightarrow a \\ \theta \rightarrow 0 \end{array} \right.$$

$$= 4 \left(a^3 - \frac{a^3}{3} \right) - 4a^2 \int_{\pi/2}^0 \left\{ \sin^{-1} \sqrt{a^2 - a^2 \cos^2 \theta} \right\} (-a \sin \theta d\theta)$$

$$= 4 \frac{2a^3}{3} - 4a^2 \int_0^{\pi/2} a \sin \theta a \sin \theta d\theta$$

$$= \frac{8a^3}{3} - 4a^4 \underbrace{[\sin 2\theta]}_0^{\pi/2}$$

$$V = \frac{8a^3}{3}$$

$$Q4 \quad V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} 1 \cdot dz \, dy \, dx$$

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$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left\{ z \right\}_{\substack{4-y \\ 0}} dy \, dx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y) dy \, dx$$

$$= \int_{-2}^2 \left[4y - \frac{y^2}{2} \right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^2 \left\{ 4\sqrt{4-x^2} - \frac{1}{2}(4-x^2) \right\} - \left\{ -4\sqrt{4-x^2} - \frac{(4-x^2)}{2} \right\} dx$$

$$= \int_{-2}^2 \left[4\sqrt{4-x^2} - \left(\frac{4-x^2}{2} \right) + 4\sqrt{4-x^2} + \left(\frac{4-x^2}{2} \right) \right] dx$$

$$\Rightarrow 8 \int_{-2}^2 \sqrt{4-x^2} \, dx$$

$$\Rightarrow 8 \int_{-2}^2 \sqrt{(2)^2 - x^2} \, dx$$

$$\Rightarrow 8 \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_{-2}^2$$

$$\Rightarrow 8 \left[(0 + 2 \sin^{-1} 1) - (0 + 2 \sin^{-1} (-1)) \right]$$

$$\Rightarrow 8 \left[2 \sin^{-1} \sin \frac{\pi}{2} + 2 \sin^{-1} \sin \frac{\pi}{2} \right]$$

$$\Rightarrow 32 \frac{\pi}{2}$$

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Sequence and Series

Fourier Series (F.S) also known as Euler Series:-

→ Let $f(x)$ be a periodic function with period 2π . Then

Fourier Series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad \dots (1)$$

① For interval $(0, 2\pi)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

② For interval $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Formula

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad ; f(x) \text{ is even function}$$

$$= 0 \quad ; f(x) \text{ is odd function}$$

Periodic function

A function $f(x)$ is said to be periodic with period T is

$$f(x) = f(x+T) = f(x+2T) = \dots$$

* To Find Period

$$(1) f(x+T) = f(x)$$

$$(2) f(x+2T) = f(x+T)$$

$$\text{Ex } (1) f(x) = f(x+2\pi)$$

$$\text{Period} = x+2\pi - x = 2\pi$$

NOTE: (1) Interval: $(0, 2\pi)$

$$\text{Periodic} = 2\pi - 0 = 2\pi$$

(2) Interval: $(-\pi, \pi)$

$$\text{Periodic} = +\pi - (-\pi) = 2\pi$$

Even Function

($f(x)$)
A function said to be even if $f(x) = f(-x)$

$$\text{Ex } (1) f(x) = x^2, x^4, x^6, \dots$$

$$\text{Put } x = -x$$

$$f(-x) = (-x)^2, (-x)^4, (-x)^6, \dots$$

$$= x^2, x^4, x^6, \dots$$

$$\Rightarrow f(-x) = f(x) //$$

$$(2) f(x) = \cos x$$

$$\text{Put } x = -x$$

$$\Rightarrow f(x) = \cos(-x)$$

$$= \cos(x)$$

$$\Rightarrow f(-x) = f(x) //$$

Odd function

($f(x)$)
A function said to be odd if $f(-x) = -f(x)$

$$\text{Ex } (1) f(x) = x, x^3, x^5, \dots$$

$$\text{Put } x = -x$$

$$f(-x) = -x, (-x)^3, (-x)^5, \dots$$

$$= -x, -x^3, -x^5, \dots$$

$$\Rightarrow f(-x) = -f(x) //$$

$$(2) f(x) = \sin x, \tan x$$

$$\text{Put } x = -x$$

$$f(-x) = \sin(-x), \tan(-x)$$

$$= -\sin x, -\tan x$$

$$\Rightarrow f(x) = -f(x) //$$

II. For Interval $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

From II

Case 1: If $f(x)$ is even

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

Even $\times$ Even = Even

$$\Rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Even $\times$ Odd = Odd

$$\Rightarrow b_n = 0$$

Case 2: If $f(x)$ is odd

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0 = a_0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

Odd $\times$ Even = Odd

$$\Rightarrow a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Odd $\times$ Odd = Even

$$\Rightarrow b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Finally if $f(x) = \text{even}$

eq (i) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (ii)}$$

↓
Half range Cosine Series

Where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

Finally if $f(x) = \text{odd}$

eqn (i) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{--- (iii)}$$

↓
Half range Sine Series

Where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

• Integration By part Rule:-

$$\boxed{\int u v dx = u \int v dx - \int (u' \int v dx) dx} \quad u, v = f(x)$$

$$\begin{aligned} \text{Ex.. } \int x \sin(2x) dx &= x \left(-\frac{\cos 2x}{2} \right) - \int 1 \left(-\frac{\cos 2x}{2} \right) dx \\ &= -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \end{aligned}$$

→ Reduction Formula

$$\boxed{\int u \cdot v dx = UV_1 - U'V_2 + U''V_3 - U'''V_4 + \dots}$$

$$\begin{aligned} \text{Ex. ① } \int x^4 \sin 2x dx &= x^4 \frac{-\cos 2x}{2} - 4x^3 \frac{-\sin 2x}{4} + 12x^2 \frac{\cos 2x}{8} \\ &\quad - 24x \frac{\sin 2x}{16} + \frac{24 \cos 2x}{32} \end{aligned}$$

$$\textcircled{\text{ii}} \int e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$\textcircled{\text{iii}} \int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

Some Formulae

- $\sin(n\pi) = 0$ | $\sin \pi, \sin 2\pi, \sin 3\pi, \dots = 0$
- $\cos(n\pi) = (-1)^n$ | $\cos \pi = \cos 3\pi = \cos 5\pi = -1$ | $\cos 2\pi = \cos 4\pi = \dots = 1$

Q Find F.S for $f(x) = e^x$ in $(-\pi, \pi)$

Solve: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \dots (1)$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} [e^x]_{-\pi}^{\pi} = \frac{2}{\pi} \left[\frac{e^{\pi} - e^{-\pi}}{2} \right]$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{e^{\pi} - e^{-\pi}}{2} \right] = \frac{2}{\pi} \sinh \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx dx$$

$$\left[\begin{array}{l} \int e^{ax} \cos bx dx \\ = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \\ a=1, b=n \end{array} \right]$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} \cos nx + n \sin nx \right]_{-\pi}^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi} \frac{e^{\pi}}{1+n^2} \{ \cos n\pi + 0 \} - \frac{e^{-\pi}}{1+n^2} \{ \cos n(-\pi) + 0 \}$$

$$\Rightarrow a_n = \frac{2 \cos n\pi}{\pi(1+n^2)} \left[\frac{e^{\pi} - e^{-\pi}}{2} \right] = \frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi$$

Also

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} \sin nx - n \cos nx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{\pi}}{1+n^2} \{ 0 - n \cos n\pi \} - \frac{e^{-\pi}}{1+n^2} \{ 0 - n \cos n(-\pi) \} \right]$$

$$\Rightarrow b_n = \frac{1}{\pi} \frac{-n e^{\pi} (-1)^n}{1+n^2} + \frac{e^{-\pi} n (-1)^n}{1+n^2}$$

$$= \frac{-2n(-1)^n}{\pi(1+n^2)} \left[\frac{e^{\pi} - e^{-\pi}}{2} \right]$$

$$= \frac{-2n(-1)^n}{\pi(1+n^2)} \sinh \pi$$

from ①

$$f(x) = e^x = \frac{1}{2} \frac{2}{\pi} \sinh \pi + \sum \frac{2(-1)^n}{\pi(1+n^2)} \sinh \pi \cos(nx) \\ + \sum \frac{-2n(-1)^n}{\pi(1+n^2)} \sinh \pi \sin(nx)$$

$$\Rightarrow e^x = \frac{1}{\pi} \sinh \pi \left[1 + \sum \frac{2(-1)^n}{1+n^2} \cos nx - \sum \frac{2n(-1)^n}{1+n^2} \sin nx \right] //$$

Q Find F.S for $f(x) = e^{-x}; (0, 2\pi)$

Soln. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ --- ①

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx = \frac{1}{\pi} [e^{-x}]_0^{2\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} [e^{-2\pi} - e^0] = \frac{1}{\pi} (e^{-2\pi} - 1)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \\ = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos(nx) dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-1 \cdot \cos nx + n \sin nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} \{-\cos n(2\pi) + n \sin n(2\pi)\} \right. \\ \left. - \frac{e^{-0}}{1+n^2} \{-\cos 0 + n \sin 0\} \right]$$

$$\Rightarrow a_n = \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} \{-\cos 2n\pi\} - \frac{1}{1+n^2} \{-\cos 0\} \right]$$

$$\Rightarrow a_n = \frac{-1}{\pi(1+n^2)} [e^{-2\pi} - 1]$$

Also,

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin(nx) dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-1 \sin(nx) - n \cos(nx)) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} \{0 - 1\} - \frac{e^0}{1+n^2} \{0 - 1\} \right]$$

$$\Rightarrow b_n = \frac{1}{\pi(1+n^2)} [e^{-2\pi} - 1]$$

from ①

$$f(x) = e^{-x} = \frac{1}{2} \left[\frac{1}{\pi} (e^{-2\pi} - 1) \right] + \sum_{n=1}^{\infty} \frac{-1}{\pi(1+n^2)} [e^{-2\pi}] \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi(1+n^2)} [e^{-2\pi} - 1] \sin(nx) //$$

$$\Rightarrow e^{-x} = \frac{-1(e^{-2\pi} - 1)}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} + \frac{\sin nx}{1+n^2} \right] //$$

Q Find F.S For $f(x) = x+x^2$ in $(-\pi, \pi)$

and hence deduced that $\pi^2/6 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

Soln:- let $f(x) = x+x^2$; $(-\pi, \pi)$

$$\text{and, } f(x) = x+x^2 = \frac{a_0}{2} + \sum a_n \cos(nx) + b_n \sin(nx) \quad \dots \textcircled{1}$$

$$\text{Where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \overset{\text{odd}}{x} dx + \int_{-\pi}^{\pi} \overset{\text{even}}{x^2} dx \right]$$

$$= \frac{1}{\pi} \left[0 + \frac{2}{1} \int_0^{\pi} x^2 dx \right]$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0 \right] = \frac{2}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \cos(nx) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \underset{0 \times E = 0}{x \cos nx} + \int_{-\pi}^{\pi} \underset{E \times E = E}{x^2 \cos nx} dx \right]$$

$$= \frac{1}{\pi} \left[0 + 2 \int_0^{\pi} x^2 \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[x^2 \frac{\sin(nx)}{n} - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(0 + \frac{2\pi \cos n\pi}{n^2} - 0 \right) - (0 + 0 - 0) \right]$$

$$\Rightarrow a_n = \frac{4(-1)^n}{n^2}$$

Also

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \sin(nx) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \underset{0 \times \theta = 0}{x \sin nx} dx + \int_{-\pi}^{\pi} \underset{\pi \times 0 = 0}{x^2 \sin nx} dx \right]$$

$$\begin{aligned}
 \Rightarrow b_n &= \frac{1}{\pi} \left[2 \int_0^{\pi} x \sin(nx) dx \right] \\
 &= \frac{2}{\pi} \left[x \left(\frac{\cos nx}{n} \right) - 1 \left(\frac{-\sin(nx)}{n} \right) \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left[\left(-\frac{\pi \cos n\pi}{n} + 0 \right) - (0 + 0) \right]
 \end{aligned}$$

$$\Rightarrow b_n = -\frac{2(-1)^n}{n}$$

from (1)

$$x+x^2 = \frac{1}{2} \cdot \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n} \sin(nx)$$

$$\Rightarrow x+x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx) \quad \text{--- (11)}$$

Now

1. Put $x=0$ in (11)

$$0+0 = \frac{\pi^2}{3} + 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\Rightarrow -\frac{\pi^2}{3} = -4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

2. Put $x=\pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$$

3. Put $x=-\pi$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

Add ② and ③

$$2\pi^2 = \frac{2\pi^2}{3} + 8 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow 2\pi^2 - \frac{2\pi^2}{3} = 8 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow \frac{4\pi^2}{3 \times 8} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

//

Q Find F.S for $f(x) = x - x^2$ in $(-\pi, \pi)$ and hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

Soln: Given $f(x) = x - x^2$

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad \text{--- (1)}$$

Where

$$\bullet a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \overset{\text{odd}}{x} dx - \int_{-\pi}^{\pi} \overset{\text{even}}{x^2} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi} x^2 dx$$

$$\Rightarrow a_0 = -\frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = -\frac{2}{\pi} \frac{\pi^3}{3} = -\frac{2}{3} \pi^2 //$$

$$\bullet a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \underset{O \times E = O}{x \cos nx} dx - \int_{-\pi}^{\pi} \underset{E \times E = E}{x^2 \cos nx} dx \right]$$

$$= \frac{1}{\pi} \left[0 - 2 \int_0^{\pi} x^2 \cos nx \, dx \right]$$

$$= -\frac{2}{\pi} \left[\frac{x^2 \sin(nx)}{n} - \frac{2x(-\cos(nx))}{n^2} + \frac{2(-\sin(nx))}{n^3} \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[\left(0 + \frac{2\pi \cos n\pi}{n^2} - 0 \right) - (0 + 0 - 0) \right]$$

$$\Rightarrow a_n = -\frac{4(-1)^n}{n^2}$$

Also.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \sin(nx) \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \sin(nx) \, dx - \int_{-\pi}^{\pi} x^2 \sin(nx) \, dx \right]$$

$$= \frac{1}{\pi} \left[2 \int_0^{\pi} x \sin(nx) \, dx - 0 \right]$$

$$= \frac{2}{\pi} \left[\frac{x(-\cos nx)}{n} - 1 \left(\frac{\sin(nx)}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left\{ \frac{\pi \cos n\pi}{n} + 0 \right\} - \left\{ 0 - 0 \right\} \right]$$

$$\Rightarrow b_n = \frac{-2 \cos(n\pi)}{n} = -\frac{2(-1)^n}{n}$$

from (i)

$$x-x^2 = \frac{1}{2} \left(\frac{-2\pi^2}{3} \right) + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx)$$

$$\Rightarrow x-x^2 = -\frac{\pi^2}{3} + (-4) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

(ii)

Now Put $x=0$ in (21)

$$0-0 = -\frac{\pi^2}{3} - 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots //$$

Q Find F.S for $f(x) = x^2$, in $(-\pi, \pi)$ and hence deduced that

$$i) \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$ii) \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$iii) \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$iv) \frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots$$

$$v) \sum \frac{1}{n^4} = \frac{\pi^4}{90} \text{ i.e. } \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

Soln:- Given $f(x) = x^2$

$$\text{Let } f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{--- (1)}$$

Where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

$$= \frac{1}{\pi} \cdot 2 \int_0^{\pi} x^2 \cos(nx) dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi} \left[\left(0 - \frac{2\pi(-\cos(n\pi))}{n^2} - 0 \right) - (0 + 0 - 0) \right]$$

$$= \frac{4(-1)^n}{n^2}$$

Also,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(n\pi) dx$$

even $\times$ odd = odd

$$\Rightarrow b_n = 0$$

from (1)

$$f(x) = x^2 = \frac{1}{2} \times \left(\frac{2}{3} \pi^2 \right) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} 0$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad \text{--- (ii)}$$

(1) Put $x = 0$ in (ii)

$$0 = \frac{\pi^2}{3} + 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\Rightarrow -\frac{\pi^2}{3} = -4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \quad \text{--- (iv)} \quad \left. \begin{array}{l} \text{1st deduction} \\ \text{---} \end{array} \right\}$$

(2) Put $x = \pi$ in (ii)

$$\pi^2 = \frac{\pi^2}{3} + 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \text{--- (v)} \quad \left. \begin{array}{l} \text{2nd deduction} \\ \text{---} \end{array} \right\}$$

(3) $\rightarrow$ (iv) + (v)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$$

$$\Rightarrow \frac{3\pi^2}{12 \times 2} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \dots \textcircled{vi} \quad \left. \vphantom{\frac{\pi^2}{8}} \right\} \text{3rd deduction}$$

$$\textcircled{4} \rightarrow \textcircled{v} - \textcircled{iv}$$

$$\frac{\pi^2}{6} - \frac{\pi^2}{12} = 2 \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{12 \times 2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots$$

$$\Rightarrow \frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \quad \textcircled{vii} \quad \left. \vphantom{\frac{\pi^2}{24}} \right\} \text{4th deduction}$$

⑤ We know that for f. cosine series
Parseval identity

$$\int_0^\pi \{f(x)\}^2 dx = \frac{\pi}{2} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

$$\int_0^\pi \{x^2\}^2 dx = \frac{\pi}{2} \left[\frac{1}{2} \frac{4\pi^4}{9} + \sum_{n=1}^{\infty} \left\{ \frac{4(-1)^n}{n^2} \right\}^2 \right]$$

$$\left[\frac{x^5}{5} \right]_0^\pi = \frac{\pi}{2} \left[\frac{2\pi^4}{9} + 16 \sum \frac{1}{n^4} \right]$$

$$\left(\frac{\pi^5}{5} - 0 \right) = \frac{\pi^5}{9} + 8\pi \sum \frac{1}{n^4}$$

$$\frac{\pi^5}{5} - \frac{\pi^5}{9} = 8\pi \sum \frac{1}{n^4}$$

$$\frac{4\pi^5}{45} = 8\pi \sum \frac{1}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4\pi^5}{8\pi \times 45}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90} \quad \text{5th Deduction}$$

Proved

Parseval Identity:

Given F.S $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

i) For interval $(-\pi, \pi)$, Parseval Identity are

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right]$$

ii) For Cosine series, $(0, \pi)$

$$\int_0^{\pi} [f(x)]^2 dx = \frac{\pi}{2} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right]$$

iii) For Sine series, $(0, \pi)$

$$\int_0^{\pi} [f(x)]^2 dx = \frac{\pi}{2} \left[\sum_{n=1}^{\infty} b_n^2 \right]$$

Q. Find $f(x) = x \sin x$ in $(-\pi, \pi)$

Soln: Let $f(x) = x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ ①

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx$$

$0 \times 0 = \text{even}$

$$= \frac{2}{\pi} [x(-\cos x) - (-\sin x)]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi} [\pi(-1) + 0] - [0 - 0]$$

$$= 2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos nx \, dx$$

$\text{even} \times \text{even} = \text{even}$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx \quad \text{②}$$

$$= \frac{2}{\pi} \int_0^{\pi} x (2 \sin x \cos nx) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right)$$

$$- \left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right) \Big|_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi} \left[\pi \left(-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right) - (0 - 0) \right]$$

$$\Rightarrow a_n = -\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}$$

$$= (-1)^n \left[\frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$\Rightarrow a_n = (-1)^n \left[\frac{n-1-n-1}{n^2-1} \right]$$

$$\Rightarrow a_n = -\frac{2(-1)^n}{n^2-1} ; n \neq 1$$

from ② put $n=1$.

$$a_1 = \frac{2}{\pi} \left[\int_0^\pi x \sin x \cos x dx \right]$$

$$= \frac{1}{\pi} \int_0^\pi x \sin 2x dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos 2x}{2} \right) - \left(\frac{-\sin 2x}{4} \right) \right]_0^\pi$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi}{2} \cos 2\pi + 0 \right) - (0+0) \right]$$

$$= \frac{1}{\pi} (-\pi/2)$$

$$\Rightarrow a_1 = -\frac{1}{2}$$

from ①

$$f(x) = x \sin x = \frac{1}{2} a_0 + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{2} \pi + \left(-\frac{1}{2} \right) \cos x + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos nx$$

$$\Rightarrow x \sin x = 1 + \frac{\cos x}{-2} + 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx$$

Q Find half range Cosine series for $\pi x - x^2$; $(0, \pi)$

Soln: Let $f(x) = \pi x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$... ①

where,

$$a_0 = \frac{2}{\pi} \int_0^\pi (\pi x - x^2) dx$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^\pi = \frac{2}{\pi} \left[\frac{\pi \pi^2}{2} - \frac{\pi^3}{3} \right] = \frac{\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx \, dx \\
&= \frac{2}{\pi} \left[\pi \int_0^{\pi} x \cos nx \, dx - \int_0^{\pi} x^2 \cos nx \, dx \right] \\
&= \frac{2}{\pi} \left[\pi \left[\left\{ \frac{x \sin nx}{n} - \frac{(1x - \cos nx)}{n^2} \right\} \right]_0^{\pi} - \left[\left\{ \frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2 \sin nx}{n^3} \right\} \right]_0^{\pi} \right] \\
&= \frac{2}{\pi} \left[\pi \left[0 + \frac{\cos n\pi}{n^2} \right] - \pi \left[0 + \frac{1}{n^2} \right] - \left[0 + \frac{2\pi \cos n\pi}{n^2} + 0 \right] - (0 + 0 - 0) \right] \\
&= \frac{2}{\pi} \left[\frac{\pi \cos n\pi}{n^2} + \frac{\pi^2}{n^2} - \frac{2\pi \cos n\pi}{n^2} \right] \\
&= \frac{2}{n^2} (1 - \cos n\pi) \\
&= \frac{2}{n^2} \{1 - (-1)^n\}
\end{aligned}$$

from ①

$$f(x) = \pi x - x^2 = \frac{1}{2} \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2\{1 - (-1)^n\}}{n^2} \cos nx$$

$$\Rightarrow \pi x - x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2\{1 - (-1)^n\}}{n^2} \cos nx$$

Q Find half range sine series for $f(x) = x; (0, \pi)$

Solⁿ:- let $f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$

where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$\begin{aligned}
 \Rightarrow b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx \\
 &= \frac{2}{\pi} \left[x \frac{(-\cos nx)}{n} - \frac{-\sin(nx)}{n^2} \right]_0^{\pi} \\
 &= \frac{2}{\pi} \left[\left\{ -\frac{\pi \cos n\pi}{n} + 0 \right\} - \{0 - 0\} \right] \\
 &= -\frac{2\pi}{\pi} \frac{\cos n\pi}{n} \\
 &= -\frac{2 \cos n\pi}{n}
 \end{aligned}$$

from (1)

$$f(x) = x = 2 \sum_{n=1}^{\infty} \frac{-\cos n\pi}{n} \sin nx$$

$$x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$$

$$\Rightarrow x = -2 \left[-1 \sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} \dots \right] //$$

Q10 Find F.S for $f(x) = |x|$ in $(-\pi, \pi)$ and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Solⁿ. let $f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ (1)

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \quad \text{even}$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) \Rightarrow a_0 = \frac{2\pi}{2} //$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} - \frac{(-\cos nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{1}{n^2}\right) \right]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \overset{\text{Even} \times \text{odd} = \text{odd}}{\sin(nx)} \, dx$$

$$= 0$$

from (1)

$$f(x) = |x| = \frac{2\pi}{2} \times \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 1}{\pi n^2} \cos nx$$

$$\Rightarrow |x| = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx \quad // (11)$$

Put $x=0$ in (11)

$$0 = \frac{\pi^2}{2} + \frac{2(-2)}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow -\frac{\pi^2}{2} = -\frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots //$$

Fourier Series (F.S)

Let $f(x)$ be a periodic function of period $2l$. The Fourier Series is defined as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

→ For interval $(0, 2l)$

$$\odot a_0 = \frac{1}{l} \int_0^{2l} f(x) dx \quad \odot a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$\odot b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$$

→ For interval $(-l, l)$

$$\odot a_0 = \frac{1}{l} \int_{-l}^l f(x) dx \quad \odot a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$\odot b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

Case 1 If $f(x)$ is even

$$\bullet a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$\bullet a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$\bullet b_n = 0$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

↓
Cosine series

Case 2 If $f(x)$ is odd

$$\bullet a_0 = a_n = 0$$

$$\bullet b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

↓
Sine series

Q Find F.S for $f(x) = x^2$ ($-l, l$)

Solve: Given $f(x) = x^2$: ($-l, l$)
even

$$\therefore b_n = 0$$

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \dots \quad (1)$$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{l} \int_0^l x^2 dx = \frac{2}{l} \left[\frac{x^3}{3} \right]_0^l = \frac{2}{l} \frac{l^3}{3} = \frac{2l^2}{3} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l x^2 \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\left\{ \frac{x^2 \sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right\} - 2x \left\{ \frac{-\cos \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right\} + 2 \left\{ \frac{-\sin \frac{n\pi x}{l}}{\frac{n^3 \pi^3}{l^3}} \right\} \right]_0^l$$

$$\Rightarrow a_n = \frac{2}{l} \left[\left\{ 0 + \frac{2l^2}{n^2 \pi^2} \cos n\pi + 0 \right\} - (0) \right]$$

$$\Rightarrow a_n = \frac{4l^2}{n^2 \pi^2} \cos n\pi = \frac{4l^2}{n^2 \pi^2} (-1)^n$$

from (1)

$$f(x) = \frac{1}{2} \left(\frac{2l^2}{3} \right) + \sum_{n=1}^{\infty} \frac{4l^2}{n^2 \pi^2} (-1)^n \cos \frac{n\pi x}{l} + 0$$

$$x^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l} //$$

$$\Rightarrow x^2 = \frac{1^2}{3} + \frac{41^2}{\pi^2} \left[-\frac{1}{1^2} \cos \frac{1\pi x}{1} + \frac{1}{2^2} \cos \frac{2\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{3} + \dots \right] \dots \textcircled{ii}$$

Put $x=0$ in $\textcircled{ii}$

$$0 = \frac{1^2}{3} + \frac{41^2}{\pi^2} \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\Rightarrow -\frac{1^2}{3} = -\frac{41^2}{\pi^2} \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \quad // \quad \textcircled{iii}$$

Put $x=1$ or $x=-1$ in $\textcircled{ii}$

$$1^2 = \frac{1^2}{3} + \frac{41^2}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow 1^2 - \frac{1^2}{3} = \frac{41^2}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad // \quad \textcircled{iv}$$

Q Find FS for $f(x) = x$ find half range cosine series in $(0,2)$.

Soln. Let cosine series

Period $(0,2)$

$\therefore l=2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$\dots \textcircled{i}$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \int_0^2 x dx = \frac{2}{2} \left[\frac{x^2}{2} \right]_0^2 = \frac{2 \times 2^2}{2} = 2$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \int_0^2 x \cos \frac{n\pi x}{2} dx$$

$$= \left[x \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{2}} \right]_0^2$$

$$\Rightarrow a_n = \left[\left\{ 0 + \frac{4}{n^2 \pi^2} \cos n\pi \right\} - \left\{ 0 + \frac{4}{n^2 \pi^2} \right\} \right]$$

$$= \frac{4}{n^2 \pi^2} (-1)^n - \frac{4}{n^2 \pi^2} = \frac{4}{n^2 \pi^2} [(-1)^n - 1]$$

$$\Rightarrow \text{If } n = \text{even}; a_n = 0$$

from ①

$$f(x) = x = 1 + \sum_{n=\text{odd}}^{\infty} \frac{4}{n^2 \pi^2} (-1)^n \cos \frac{n\pi x}{2}$$

$$f(x) = x = 1 + \frac{4(-2)}{\pi^2} \left[\frac{\cos \frac{\pi}{2} x}{1^2} + \frac{\cos \frac{3\pi}{2} x}{3^2} + \frac{\cos \frac{5\pi}{2} x}{5^2} + \dots \right]$$

∴ Develop $\sin \frac{\pi x}{d}$ in half range cosine series in $(0, 1)$.

Soln: Let half range cosine series of $f(x) = \sin \frac{\pi x}{d}$

$$= f(x) = \sin \frac{\pi x}{d} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{d}$$

Where,

$$a_0 = \frac{2}{d} \int_0^d \sin \frac{\pi x}{d} dx$$

$$= \frac{2}{d} \left[\frac{\cos \frac{\pi x}{d}}{\pi/d} \right]_0^d$$

$$= -\frac{2}{\pi} (\cos \pi - \cos 0)$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{d} \int_0^d \sin \frac{\pi x}{d} \cos \frac{n\pi x}{d} dx \quad \text{--- (ii)}$$

$$= \frac{2}{d} \cdot \frac{1}{2} \left[\int_0^d \left(\sin \frac{(1+n)\pi x}{d} + \sin \frac{(1-n)\pi x}{d} \right) dx \right]$$

$$= \frac{1}{d} \left[\frac{-\cos \frac{(1+n)\pi x}{d}}{\frac{(1+n)\pi}{d}} + \frac{-\cos \frac{(1-n)\pi x}{d}}{\frac{(1-n)\pi}{d}} \right]_0^d$$

$$\Rightarrow a_n = \frac{1}{4} \left[\sqrt{\frac{\cos(1+n)\pi}{(1+n)\pi/4} - \frac{\cos(1-n)\pi}{(1-n)\pi/4}} \right. \\ \left. - \left[\frac{1}{(1+n)\pi/4} - \frac{1}{(1-n)\pi/4} \right] \right]$$

$$\Rightarrow a_n = \frac{1}{4} \left(\frac{1}{\pi} \right) \left[\frac{-(-1)^{n+1}}{(n+1)} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{n+1} + \frac{1}{1-n} \right] \\ = \frac{1}{\pi} \left[\frac{1-(-1)^{n+1}}{(n+1)} + \frac{1-(-1)^{1-n}}{(1-n)} \right]$$

at $n=1$
not defined

from (i) at $n=1$

$$a_1 = \frac{2}{\pi} \int_0^1 \sin \frac{\pi x}{4} \cos \frac{\pi x}{4} dx =$$

$$a_1 = \frac{1}{4} \int_0^1 2 \sin \frac{\pi x}{4} \cos \frac{\pi x}{4} dx$$

$$= \frac{1}{4} \int_0^1 \sin \frac{2\pi x}{4} dx$$

$$= \frac{1}{4} \left[-\cos \frac{2\pi x}{4} \right]_0^1$$

$$a_1 = \frac{1}{4} \left[-\frac{1}{2\pi} - \frac{(-1)}{2\pi} \right] = 0$$

Now

from (i)

$$f(x) = \frac{2}{\pi} + \frac{1}{\pi} \sum_{n=2}^{\infty} \left(\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{1-n}}{1-n} \right) \cos \frac{n\pi x}{4}$$

Q Find the F.S for periodic function

$$f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}$$

Soln

Let F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{--- (1)}$$

where

$$\bullet a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) dx + \int_0^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi dx + \int_0^{\pi} x dx \right]$$

$$= \frac{1}{\pi} \left[-[\pi x]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\{0 - (-\pi^2)\} + \left\{0 - \frac{\pi^2}{2}\right\} \right]$$

$$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right]$$

$$\Rightarrow a_0 = \frac{-3\pi^2}{2\pi} = -\frac{3}{2}\pi$$

$$\bullet a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos nx dx + \int_0^{\pi} x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[-\pi \left\{ \frac{\sin nx}{n} \right\}_{-\pi}^0 + \left\{ x \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\pi (0 - 0) + \left\{ (0 + \frac{\cos n\pi}{n^2}) - (0 + \frac{1}{n^2}) \right\} \right]$$

$$\Rightarrow a_n = \frac{[(-1)^n - 1]}{\pi n^2}$$

$$\begin{aligned}
 \bullet \quad b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin nx \, dx + \int_0^{\pi} x \sin nx \, dx \right] \\
 &= \frac{1}{\pi} \left[+\pi \left\{ \frac{\cos nx}{n} \right\}_{-\pi}^0 + \left\{ x \left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right\}_0^{\pi} \right] \\
 &= \frac{1}{\pi} \left[\pi \left(\frac{1}{n} - \frac{\cos n\pi}{n} \right) + \left(\pi \left(-\frac{\cos n\pi}{n} \right) + 0 \right) - (0 + 0) \right] \\
 &= \frac{\pi}{\pi n} [1 - \cos n\pi + (-\cos n\pi)] \\
 &= \frac{1}{n} [1 - 2(-1)^n]
 \end{aligned}$$

from (1)

$$\begin{aligned}
 f(x) &= -\frac{3\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{\pi n^2} \cos nx + \frac{1-2(-1)^n}{n} \sin nx \right] \\
 &= -\frac{3\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{1-2(-1)^n}{n} \sin nx //
 \end{aligned}$$

Q Find the F.S for $|\sin x|$ in $(-\pi, \pi)$

Solve let $f(x) = |\sin x|$

Put $x = -x$

$$f(-x) = |\sin(-x)| = |-\sin x|$$

$$f(-x) = \sin x = |\sin x| = f(x)$$

$\therefore f(x) = |\sin x|$ is even

$$\Rightarrow \underline{b_n = 0}$$

We have fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx = \frac{2}{\pi} \int_0^{\pi} |\sin x| \, dx$$

$$\Rightarrow a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{2}{\pi} [-\cos x]_0^{\pi} = -\frac{2}{\pi} [-1 - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$\Rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos nx \, dx \quad (2)$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \cos(nx) \sin x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} - \left\{ -\frac{\cos(n-1)x}{n-1} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\left\{ \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} \right\} - \left\{ \frac{1}{n+1} - \frac{1}{n-1} \right\} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n (-1)}{n+1} - \frac{(-1)^n (-1)^{-1}}{n-1} - \left\{ \frac{n-1-n-1}{n^2-1} \right\} \right]$$

$$= \frac{1}{\pi} \left[(-1)^n \left\{ -\frac{1}{n+1} + \frac{1}{n-1} \right\} + \frac{2}{n^2-1} \right]$$

$$= \frac{1}{\pi} \left[\frac{2(-1)^n}{n^2-1} + \frac{2}{n^2-1} \right]$$

$$\Rightarrow a_n = \frac{-2}{\pi(n^2-1)} [(-1)^n + 1]$$

Here, $n=1$ so it is not defined

from (2) put $n=1$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \int_0^{\pi} 2 \sin x \cos x \, dx$$

$$= \int_0^{\pi} \sin(2x) \, dx$$

$$\Rightarrow a_1 = \frac{1}{\pi} \left[\frac{\cos 2n}{2} \right]_0^\pi = \frac{1}{2\pi} [\cos 2\pi - \cos 0]$$

$$\Rightarrow a_1 = \frac{1}{2\pi} [1-1] = 0 //$$

$$\bullet b_n = 0$$

from (1)

$$f(x) = |\sin x| = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-2((-1)^n + 1)}{\pi(n^2 - 1)} \cos nx$$

$$|\sin x| = \frac{2}{\pi} + \left(-\frac{2}{\pi}\right) \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{(n^2 - 1)} \cos nx //$$

Q Obtain the half range Cosine series for

$$f(x) = \begin{cases} kx & ; 0 \leq x \leq \frac{1}{2} \\ k(1-x) & ; \frac{1}{2} \leq x \leq 1 \end{cases}$$

Solⁿ: let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1} \dots (1)$

where,

$$\bullet a_0 = \frac{2}{1} \int_0^1 f(x) dx$$

$$\Rightarrow a_0 = \frac{2}{1} \left[\int_0^{\frac{1}{2}} kx dx + \int_{\frac{1}{2}}^1 k(1-x) dx \right]$$

$$\Rightarrow a_0 = \frac{2}{1} \left[k \left\{ \frac{x^2}{2} \right\}_0^{\frac{1}{2}} + k \left\{ \frac{(1-x)^2}{-2} \right\}_{\frac{1}{2}}^1 \right]$$

$$\Rightarrow a_0 = \frac{2}{1} \left[\left\{ k \left(\frac{1}{8} - 0 \right) \right\} - \left\{ k \left(0 - \frac{1}{8} \right) \right\} \right]$$

$$= \frac{2}{1} \left[\frac{k}{8} + \frac{k}{8} \right]$$

$$\Rightarrow a_0 = \frac{k}{2}$$

$$\begin{aligned}
 a_n &= \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi}{l} x dx \\
 &= \frac{2}{l} \left[\int_0^{l/2} kx \cos \frac{n\pi}{l} x dx + \int_{l/2}^l k(l-x) \cos \frac{n\pi}{l} x dx \right] \\
 &= \frac{2k}{l} \left[\int_0^{l/2} x \cos \frac{n\pi}{l} x dx + \int_{l/2}^l (l-x) \cos \frac{n\pi}{l} x dx \right] \\
 &= \frac{2k}{l} \left[\left\{ x \frac{\sin \frac{n\pi}{l} x}{\frac{n\pi}{l}} - \frac{1(-\cos \frac{n\pi}{l} x)}{\frac{n^2 \pi^2}{l^2}} \right\} \right]_0^{l/2} \\
 &\quad + \frac{2k}{l} \left[(l-x) \frac{(-\sin \frac{n\pi}{l} x)}{\frac{n\pi}{l}} - (-1) \frac{(-\cos \frac{n\pi}{l} x)}{\frac{n^2 \pi^2}{l^2}} \right]_{l/2}^l
 \end{aligned}$$

Q Find the F.S for $x \cos x$ in $(-\pi, \pi)$

Solⁿ: Let $f(x) = x \cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ — (1)

when,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cdot 2 \cos x \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(1-n)x] \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \sin(n+1)x \, dx - \int_0^{\pi} x \sin(1-n)x \, dx \right]$$

$$= \frac{1}{\pi} \left[\left\{ x \left[-\frac{\cos(n+1)x}{n+1} - \frac{\sin(n+1)x}{(n+1)^2} \right] \right\}_0^{\pi} - \left\{ x \left[-\frac{\cos(1-n)x}{1-n} - \frac{\sin(1-n)x}{(1-n)^2} \right] \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\left\{ \pi \left(-\frac{\cos(n+1)\pi}{n+1} + 0 \right) \right\} - \left\{ 0 + 0 \right\} - \left\{ \left(\pi \left(-\frac{\cos(1-n)\pi}{1-n} + 0 \right) \right) - (0 + 0) \right\} \right]$$

$$= \frac{\pi}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(1-n)\pi}{1-n} \right]$$

$$= \left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{1+n}}{-1+n} \right]$$

$$= - \left[\frac{(-1)(-1)^{+n}}{n+1} + \frac{(-1)^1(-1)^n}{-1+n} \right]$$

$$= +(-1)^n \left[\frac{-1(n+1) + n+1}{-(1-n^2)} \right]$$

$$\Rightarrow b_n = \frac{2(-1)^n}{-(1-n^2)}$$

at $n=1$ not define

at $n=1$; $b_n = -1/2$

from (i)

$$f(x) = x \cos x = \left\{ \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \sin nx \right\} + \left\{ -\frac{1}{2} \sin x \right\}$$

Infinite Series

Let $u_1, u_2, u_3, u_4, \dots \infty$ (sequence)

then,

$u_1 + u_2 + u_3 + \dots \infty$ (series)

nth term

$$S_n = \sum_{i=1}^n u_i = u_1 + u_2 + u_3 + \dots \quad (\text{series})$$

↓

Sum of nth terms

$$S_n = \sum u_n$$

u_n is nth term

$$\text{Ex 1. } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} + \dots \infty$$

$$u_n = \frac{1}{n^2}$$

$$\sum u_n = S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$$

To test the convergency :

i) Convergent $\rightarrow \lim_{n \rightarrow \infty} \sum u_n = \lim_{n \rightarrow \infty} S_n = \text{finite value}$

2) Divergent : $\lim_{n \rightarrow \infty} \sum u_n = \lim_{n \rightarrow \infty} S_n = +\infty$

3) Oscillatory : Not a finite or not $\pm \infty$

Ex - $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \dots \infty$

$S_2 = 1$

$S_3 = 0$

$S_5 = 5$

$S_8 = 1 \dots$

Q Show the series $\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \frac{1}{15 \cdot 19} + \dots$

$= \sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n+3)}$

Soln - let $u_n = \frac{1}{(4n-1)(4n+3)}$

$u_n = \frac{1}{4} \left[\frac{1}{4n-1} - \frac{1}{4n+3} \right]$

$\sum u_n = \frac{1}{4} \sum \left(\frac{1}{4n-1} - \frac{1}{4n+3} \right)$

$= \frac{1}{4} \left[\left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{11} \right) + \left(\frac{1}{11} - \frac{1}{15} \right) + \dots + \left(\frac{1}{4n-1} - \frac{1}{4n+3} \right) \right]$

$$\Rightarrow S_n = \sum u_n = \frac{1}{4} \left[\left(\frac{1}{3} - \frac{1}{4n+3} \right) \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{4} \left[\left(\frac{1}{3} - \frac{1}{4n+3} \right) \right]$$

$$= \frac{1}{12} = \text{finite value}$$

Hence, Series is Convergent.

Q Show that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

is convergent and its sum is $\frac{1}{2}$.

Sol. let $u_n = \frac{1}{(2n-1)(2n+1)}$

$$u_n = \frac{1}{2} \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right]$$

$$\sum u_n = \frac{1}{2} \sum \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right]$$

$$\Rightarrow S_n = \sum u_n = \frac{1}{2} \left(1 - \frac{1}{2n+3} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{2n+3} \right)$$

$$= \frac{1}{2} \quad \text{proved}$$

To Test Convergency by Theorem:-

In Order that the infinite series $\sum u_n$ is convergent it is necessary but not sufficient that

$$\lim_{n \rightarrow \infty} u_n = 0$$

if $\neq 0$, then series is divergent

Q Show that the following series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^n \text{ is divergent.}$$

Solⁿ: Given series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^n$

$$\text{let } u_n = \left(1 + \frac{1}{n} \right)^n$$

$$\therefore \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e \neq 0$$

$$= 2 < e < 3$$

$\therefore$ Series is divergent

$$\left[\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \right]$$

Q.2 Test Convergency of the series

$$\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{65}} + \dots + \frac{2^n}{\sqrt{4^n + 1}} + \dots \infty$$

Solve: $u_n = \frac{2^n}{\sqrt{4^n + 1}} = \frac{2^n}{\sqrt{(2^n)^2 \left(1 + \frac{1}{4^n}\right)}} = \frac{2^n}{2^n \sqrt{1 + \frac{1}{4^n}}}$

$$\Rightarrow u_n = \frac{1}{\sqrt{1 + \frac{1}{4^n}}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{4^n}}} = \frac{1}{\sqrt{1}} \neq 0$$

Hence, Given series is ^{not} convergent.

Q.3. Prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots \infty$ is convergent.

Sol. let, $u_n = \frac{1}{n^2} \Rightarrow \sum u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \infty = S_n$

$\therefore \lim_{n \rightarrow \infty} S_n < 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

$$\therefore \left(\frac{1}{2^2} < \frac{1}{1 \cdot 2}, \frac{1}{3^2} < \frac{1}{2 \cdot 3}, \frac{1}{4^2} < \frac{1}{3 \cdot 4} \right)$$

$$\Rightarrow S_n < 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots$$

$$= \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$\Rightarrow S_n < \left(2 - \frac{1}{n} \right)$$

$$\therefore \lim_{n \rightarrow \infty} S_n < \lim_{n \rightarrow \infty} \left(2 - \frac{1}{n} \right)$$

$$< 2 \rightarrow \text{finite value}$$

$\therefore S_n$ is convergent.

Q. Find the sum of the series

$$1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \frac{1+2+3+4}{4} + \dots$$

Solⁿ let $S_n = \sum U_n = 1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \dots + \infty$

$$\Rightarrow U_n = \frac{1+2+3+4+\dots+n}{n}$$

$$\Rightarrow U_n = \frac{n(n+1)}{2n}$$

$$= \frac{n(n+1)}{n(n-1)} = \frac{1}{2} \left[\frac{n}{n-1} + \frac{1}{n-1} \right]$$

$$\Rightarrow U_n = \frac{1}{2} \left[\frac{n-1+1}{n-1} + \frac{1}{n-1} \right]$$

$$= \frac{1}{2} \left[\frac{n-1}{(n-1)(n-1)!} + \frac{1}{(n-1)!} + \frac{1}{(n-1)!} \right]$$

$$= \frac{1}{2} \left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \right]$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!} \right]$$

$$= \frac{1}{2} \left[\frac{1}{0!} \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \right) \right]$$

$$= \frac{1}{2} \left[\frac{1}{0!} \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) \right]$$

$$\Rightarrow S_n = \frac{1}{2} [e + 2e] = \frac{3e}{2} \quad (2 < e < 3)$$

= finite value

Series is Convergent

$$\left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right]$$

$$\left[e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right]$$

Comparison Test:

If $\sum u_n$ and $\sum v_n$ are two series of the numbers. Then

1) If $\sum v_n$ is convergent Then $\sum u_n$ is also convergent such that

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \text{finite non zero quantity.}$$

ii) If $\sum V_n$ is divergent then summation of $\sum U_n$ is also divergent if

$$\lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \text{finite non zero quantity,}$$

NOTE:- $\sum V_n$ convergency depends on p-series (Auxiliary series test).

The Infinite Series

$$\sum \frac{1}{n^p} = \frac{1^p}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

is convergent if $p > 1$
and divergent if $p \leq 1$.

[V_n 's selected from U_n such that $\frac{1}{n^p}$]

Q Test the convergency of the series

$$\sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$$

Solution:- $U_n = [\sqrt{n^2+1} - n]$

$$= \left[\sqrt{n^2 \left(1 + \frac{1}{n^2}\right)} - n \right] = n \left[\left(1 + \frac{1}{n^2}\right)^{\frac{1}{2}} - 1 \right]$$

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + 1$$

OR

$$(1-x)^n = 1 - \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$\Rightarrow U_n = n \left[\left(1 + \frac{1}{2} \cdot \frac{1}{n^2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} \cdot \frac{1}{n^4} + \dots \right) \right]$$

$$= n \left[\frac{1}{2} \cdot \frac{1}{n^2} + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} \cdot \frac{1}{n^4} + \dots \right]$$

$$= \frac{1}{n} \left[\frac{1}{2} - \frac{1}{8n^2} + \dots \right]$$

(c) $V_n = \frac{1}{n}$ $\therefore \sum V_n = \sum \frac{1}{n}, p=1$

V_n is divergent

Now,

$$\lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{2} - \frac{1}{8n^2} + \dots \right]$$

$$\frac{1}{n}$$

$$= \frac{1}{2} \text{ finite non zero value}$$

Since, $\sum v_n p_n$ is divergent.

$\therefore \sum u_n p_n$ is also divergent.

Ex (11) Test the convergency of

$$U_n = \sqrt[3]{n^3+1} - n$$

$$= \left[(n^3+1)^{1/3} - n \right]$$

$$= \left[n \left(1 + \frac{1}{n^3} \right)^{1/3} - n \right]$$

$$= n \left[\left(1 + \frac{1}{n^3} \right)^{1/3} - 1 \right]$$

$$\left[(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \right]$$

$$\left[(1-x)^n = 1 - \frac{nx}{1!} + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots \right]$$

$$= n \left[\left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}(-\frac{2}{3})}{1 \times 2} \frac{1}{n^6} + \dots \right] - 1 \right]$$

$$= \frac{n}{n^3} \left[\frac{1}{3} + \frac{\frac{1}{3}(-\frac{2}{3})}{2} \frac{1}{n^3} + \dots \right]$$

$$= \frac{1}{n^2} \left[\frac{1}{3} + \frac{(-\frac{2}{9})}{2n^3} + \dots \right]$$

let $V_n = \frac{1}{n^2}$ $\therefore \sum V_n = \sum \frac{1}{n^2}$; Here $p=2$

$\lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} \left[\frac{1}{3} + \frac{(-\frac{2}{9})}{2n^3} + \dots \right]}{\frac{1}{n^2}} = \frac{1}{3}$ finite non zero quantity.

$\therefore \sum V_n$ is convergent so
 $\sum U_n$ is also convergent.

Q Test the convergency of $\frac{1}{n} \sin \frac{1}{n}$

solue. let $U_n = \frac{1}{n} \sin \frac{1}{n}$

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$U_n = \frac{1}{n} \left[\frac{1}{n} - \frac{1}{3! n^3} + \frac{1}{5! n^5} + \dots \right]$$

$$U_n = \frac{1}{n^2} \left[1 - \frac{1}{3! n^2} + \frac{1}{5! n^4} + \dots \right]$$

$$\text{let } V_n = \frac{1}{n^2} \quad \therefore \sum V_n = \sum \frac{1}{n^2} \quad ; \text{ Here } p=2 > 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} \left[1 - \frac{1}{3! n^2} + \frac{1}{5! n^4} + \dots \right]}{\frac{1}{n^2}}$$

$= 1 = \text{finite non zero value.}$

$\therefore$ Hence, $\sum V_n$ is convergent so, $\sum U_n$ is also convergent.

Q) Test the convergence of the series

$$\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \dots$$

Solues- n^{th} term

$$U_n = \frac{1}{(2n-1)^p} = \frac{1}{n^p} \left[\frac{1}{\left(2 - \frac{1}{n}\right)^p} \right]$$

Let $V_n = \frac{1}{n^p}$ } if $p > 1$, then $\sum V_n$ is convergent
 $\sum V_n = \sum \frac{1}{n^p}$ } if $p \leq 1$, then $\sum V_n$ is divergent

$$\therefore \lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^p} \left(\frac{1}{\left(2 - \frac{1}{n}\right)^p} \right)}{\frac{1}{n^p}} = \frac{1}{\left(2 - \frac{1}{\infty}\right)^p} = \frac{1}{2^p}$$

$\therefore p = 1, 2, 3$, we get non zero finite value.

$\therefore$ Since

If $\sum V_n$ is convergent then $\sum U_n$ is convergent if $p > 1$

If $\sum V_n$ is divergent then $\sum U_n$ is divergent if $p \leq 1$

Q. Test the Convergency

$$1 + \frac{1}{2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \dots$$

Sol. let
nth term

neglecting the 1st term

$$i.e. U_n = \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{n^n}{(n+1)^n (n+1)}$$

$$= \frac{n^n}{n^n \left(1 + \frac{1}{n}\right)^n (n+1)}$$

$$= \frac{1}{n} \left[\frac{1}{\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)} \right]$$

Take

$$V_n = \frac{1}{n} \quad \left\{ \begin{array}{l} \text{Here } p = 1 \leq 1 \\ \therefore \sum V_n \text{ is divergent.} \end{array} \right.$$

$$\therefore \lim_{n \rightarrow \infty} \frac{U_n}{V_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)}}{\frac{1}{n}} \quad \left\{ \begin{array}{l} \therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \\ = e \end{array} \right.$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$= \frac{1}{e} \times \frac{1}{1} = \frac{1}{e} \text{ finite non zero quantity}$$

∴ $\sum v_n$ is divergent, $\sum u_n$ is also divergent.

Cauchy's Root Test :-

→ It is defined as

i) The series $\sum u_n$ is convergent

$$\text{if } \lim_{n \rightarrow \infty} \sqrt[n]{u_n} < 1$$

ii) $\sum u_n$ is divergent

$$\text{if } \lim_{n \rightarrow \infty} \sqrt[n]{u_n} > 1$$

iii) The test fails if $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = 1$

Example Question:- The convergency of the series whose n^{th} term is $\frac{n^{n^2}}{(1+n)^{n^2}}$

Sol. Given

$$u_n = \frac{n^{n^2}}{(1+n)^{n^2}} = \left[\frac{n}{1+n} \right]^{n^2} = \left[\frac{n}{n(1+\frac{1}{n})} \right]^{n^2}$$

$$u_n = \left(\frac{1}{1+\frac{1}{n}} \right)^{n^2}$$

$$\therefore \lim_{n \rightarrow \infty} \{U_n\}^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left[\frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}} \right]^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

Therefore $\sum U_n$ is convergent.

Q Test the convergency

$$\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \dots + \frac{n^3}{3^n}$$

Sol. let $U_n = \frac{n^3}{3^n} \therefore \sum U_n = \sum \frac{n^3}{3^n}$

$$\lim_{n \rightarrow \infty} \{U_n\}^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left[\frac{n^3}{3^n} \right]^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{\frac{3}{n}}}{3}$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} n^{\frac{3}{n}}$$

$$= \frac{1}{3} < 1$$

$\therefore \sum U_n$ is convergent.

Q Test the Convergence of the Series

i) $\sum \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n} \rightarrow$ To do by yourself

ii) $\sum \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n} \right]^{-n}$

Solⁿ: $U_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n} \right]^{-n}$

$$= \left(\frac{n+1}{n}\right)^{-n} \left[\left(\frac{n+1}{n}\right)^n - 1 \right]^{-n}$$

$$\lim_{n \rightarrow \infty} \{U_n\}^{\frac{1}{n}} = \left(\frac{n+1}{n}\right)^{-1} \lim_{n \rightarrow \infty} \left\{ \left(\frac{n+1}{n}\right)^n \right\}^{\frac{1}{n}} \left\{ \left[\left(\frac{n+1}{n}\right)^n - 1 \right] \right\}^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{-1} \left\{ \left(\frac{n+1}{n}\right)^n - 1 \right\}^{-1}$$

$$= \lim_{n \rightarrow \infty} \left\{ \frac{\left(1 + \frac{1}{n}\right)^n}{n} \right\}^{-1} \left\{ \frac{\left(1 + \frac{1}{n}\right)^n}{n} - 1 \right\}^{-1}$$

$$= (1-0)^{-1} [e-1]^{-1}$$

$$= \frac{1}{e-1} < 1, \quad 2 < e < 3$$

$\therefore U_n$ is convergent.

D'Alembert's Ratio Test.

→ The series $\sum u_n$ of a term. Then it is

i) Convergent, if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} < 1$

ii) Divergent, if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} > 1$

iii) test fail, if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$

Example Questions Test the convergency $\frac{1+2^p}{2!}$,

$$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots$$

Sol.:

$$\text{Let } u_n = \frac{n^p}{n!}$$

$$\therefore u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \frac{n^p \left(1 + \frac{1}{n}\right)^p}{(n+1) n!} \times \frac{n!}{n^p}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\left(1 + \frac{1}{n}\right)^p}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{n})^p}{n+1} = \frac{1^p}{\infty} = 0 < 1$$

Hence, $\sum U_n$ is convergent.

Q Test the convergency of the series.

$$\sum \left[\frac{2^n - 1}{3^n - 1} \right]^{\frac{1}{2}}$$

Soln:- let $U_n = \left[\frac{2^n - 1}{3^n - 1} \right]^{\frac{1}{2}} \Rightarrow U_{n+1} = \left[\frac{2^{n+1} - 1}{3^{n+1} - 1} \right]^{\frac{1}{2}}$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \left[\frac{2^{n+1} - 1}{3^{n+1} - 1} \times \frac{3^n - 1}{2^n - 1} \right]^{\frac{1}{2}}$$

$$= \left[\frac{2^{n+1} \left[1 - \frac{1}{2^{n+1}} \right]}{3^{n+1} \left[1 - \frac{1}{3^{n+1}} \right]} \times \frac{3^n \left[1 - \frac{1}{3^n} \right]}{2^n \left[1 - \frac{1}{2^n} \right]} \right]^{\frac{1}{2}}$$

$$= \sqrt{\left(\frac{2}{3} \right) \left(\frac{2}{3} \right) \left(\frac{3}{2} \right)^n \left[\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{3^{n+1}}} \times \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{2^n}} \right]^{\frac{1}{2}}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \lim_{n \rightarrow \infty} \sqrt{\frac{2}{3}} \sqrt{\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{3^{n+1}}} \times \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{2^n}}}$$

$$= \sqrt{\frac{2}{3}} \times 1 = \sqrt{\frac{2}{3}} < 1$$

Hence $\sum U_n$ is convergent.

☞ Test the Convergence

$$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots$$

Solⁿ let $U_n = n^2 x^{n-1}$

$$U_{n+1} = (n+1)^2 x^{n+1-1}$$

$$= (n+1)^2 x^n$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{(n+1)^2 x^n}{n^2 x^{n-1}}$$

$$= \left(\frac{n+1}{n}\right)^2 x^{n-n+1}$$

$$= \left(\frac{n+1}{n}\right)^2 x$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \left(1 + \frac{1}{n}\right)^2 x$$

$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 x$$

$$\text{let } x = 1$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 1$$

fail. //

Vector Space

Let $(F, +, \cdot)$ be a field whose elements are called a scalar and let U be a non-zero empty set whose elements are called a vector.

Then V is called a vector space over the field F , [It is denoted by V_F or $V(F)$] if the following condition satisfies.

I. V is an abelian wrt addition. $\oplus$

i) Closure Law :- Let $\alpha, \beta \in V$

$$\Rightarrow \alpha + \beta \in V ; \forall \alpha, \beta \in V$$

• $\in \rightarrow$ belong to

• $\forall \rightarrow$ for all

$\hookrightarrow$ for every

ii) Associative Law :- Let $\alpha, \beta, \gamma \in V$

$$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) ; \forall \alpha, \beta, \gamma \in V$$

iii) Commutative Law :- Let $\alpha, \beta \in V$

$$\Rightarrow \alpha + \beta = \beta + \alpha ; \forall \alpha, \beta \in V$$

iv) Additive identity :- Let $\alpha \in V$ $\exists 0 \in V$

$$\Rightarrow \alpha + 0 = 0 + \alpha = \alpha ; \forall \alpha \in V$$

• $\exists \rightarrow$ then their exist

v) (Inverse) Additive Inverse :- Let $\alpha \in V$ $\exists (-\alpha) \in V$

$$\Rightarrow \alpha + (-\alpha) = (-\alpha) + \alpha = 0 ; \forall \alpha \in V$$

then V is called an abelian group .]

II. $\times$ Multiplication composition with scalar element over the field F

$$[\text{Let } a \in F, \alpha \in V \Rightarrow a\alpha \in V]$$

III. > Also some following cond.ⁿ are satisfied

i) $a(\alpha + \beta) = a\alpha + a\beta$; $\forall a \in F; \alpha, \beta \in V$

ii) $(a+b)\alpha = a\alpha + b\alpha$; $\forall a, b \in F; \alpha \in V$

iii) $(a \cdot b)\alpha = a(b\alpha)$; $\forall a, b \in F; \alpha \in V$

Subspace :- Let V_F be a vector space over the field F or $(F, +, \cdot)$ and Let $W \subseteq V$ ($V =$ non empty set)
Then, W or W_F is called Subspace of V_F if W itself a vector space over the field $(F, +, \cdot)$ w.r.t scalar multiplication and addition.

Linear Combination of Vector :-

Let V_F be a vector space over the field $(F, +, \cdot)$ or F

and $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\} \in V$ and $a = \{a_1, a_2, a_3, \dots, a_n\} \in F$

Then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \dots + a_n\alpha_n$$

Example : If vector $\alpha = (5, 9)$ and $\alpha_1 = (2, 3)$, $\alpha_2 = (5, 6)$, then represent α as a linear combination.

Solⁿ:- Let $a_1, a_2 \in F$

Given, $\alpha_1 = (2, 3)$ $\alpha = (5, 9)$
 $\alpha_2 = (5, 6)$

By linear combination.

$$\alpha = a_1\alpha_1 + a_2\alpha_2$$

$$\alpha = a_1(2, 3) + a_2(5, 6) \dots \text{--- (1)}$$

$$\Rightarrow (5, 9) = (2a_1, 3a_1) + (5a_2, 6a_2)$$

$$= \{(2a_1 + 5a_2), (3a_1 + 6a_2)\}$$

On Comparing both side

$$2a_1 + 5a_2 = 5 \quad \text{--- (i)}$$

$$3a_1 + 6a_2 = 9 \quad \text{--- (ii)}$$

$$\Rightarrow a_1 = 3 - 2a_2 \quad \text{--- (iv)}$$

Put in (i)

$$2(3 - 2a_2) + 5a_2 = 5$$

$$\Rightarrow -4a_2 + 5a_2 + 6 = 5$$

$$\Rightarrow \boxed{a_2 = -1}$$

Put in (iv)

$$a_1 = 3 - 2(-1)$$

$$\boxed{a_1 = 5}$$

∴ From (i)

$$\alpha = 5(2, 3) + (-1)(5, 6)$$

Imp

Linearly Dependent and Linearly Independent
(L.D) Vectors. (L.I)

i) Linearly Dependent Vectors:-

Let V_F be a vector space over the field F .

Let $S = \{a_1, a_2, a_3, a_4, \dots, a_n\} \in F$, Then S is called L.D, If there exist scalar $a_1, a_2, a_3, \dots, a_n \in F$ not all are zero, some may be zero.

Such that

$$a_1 a_1 + a_2 a_2 + a_3 a_3 + \dots + a_n a_n = 0$$

ii) Linearly Independent Vectors:-

Let V_F be a vector space over the field F .

Let $S = \{a_1, a_2, a_3, \dots, a_n\} \in F$, Then S is called

L.I, If there exist scalar $a_1, a_2, a_3, \dots, a_n \in F$, all are zero.

And

$$a_1 a_1 + a_2 a_2 + a_3 a_3 + \dots + a_n a_n = 0$$

Q Are the vectors $(2, 2, 2, 4)$, $(2, -2, -4, 0)$, $(4, -2, -5, 2)$ and $(4, 2, 1, 6)$ L.I or not.

Soln: Let $\alpha_1 = (2, 2, 2, 4)$ Let a_1, a_2, a_3 and $a_4 \in F$
 $\alpha_2 = (2, -2, -4, 0)$ Such that
 $\alpha_3 = (4, -2, -5, 2)$ $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$
 $\alpha_4 = (4, 2, 1, 6)$

$$\text{ie;} \quad a_1(2, 2, 2, 4) + a_2(2, -2, -4, 0) + a_3(4, -2, -5, 2) + a_4(4, 2, 1, 6) = 0$$

$$\Rightarrow (2a_1, 2a_1, 2a_1, 4a_1) + (2a_2, -2a_2, -4a_2, 0) + (4a_3, -2a_3, -5a_3, 2a_3) + (4a_4, 2a_4, a_4, 6a_4) = 0$$

On Comparing both side.

$$2a_1 + 2a_2 + 4a_3 + 4a_4 = 0 \quad \text{--- (i)}$$

$$2a_1 + (-2)a_2 - 2a_3 + 2a_4 = 0 \quad \text{--- (ii)}$$

$$2a_1 - 4a_2 - 5a_3 + a_4 = 0 \quad \text{--- (iii)}$$

$$4a_1 + 0a_2 + 2a_3 + 6a_4 = 0 \quad \text{--- (iv)}$$

Coefficient matrix

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & -2 & -2 & 2 \\ 2 & -4 & -5 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix}$$

[NOTE: no. of non-zero soln
 $= n - \rho(A)$
 $\downarrow$
 no. of unknown]

$$R_2 \rightarrow R_2 - R_1 \mid R_3 \rightarrow R_3 - R_1 \mid R_4 \rightarrow R_4 - 2R_1$$

$$= \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -2 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

$$R_1 \rightarrow \frac{1}{2}R_1 \mid R_2 \rightarrow \frac{1}{-2}R_2 \mid R_3 \rightarrow \frac{1}{-3}R_3 \mid R_4 \rightarrow \frac{1}{-2}R_4$$

$$= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_2 \mid R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2 < n = 4$$

Now,

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

gives non zero infinite soln.

$$a_1 + a_2 + 2a_3 + 2a_4 = 0 \quad \text{--- (v)}$$

$$2a_2 + 3a_3 + a_4 = 0 \quad \text{--- (vi)}$$

$$\text{Let } \boxed{a_2 = k_1} \mid \boxed{a_3 = k_2}$$

$$k_1, k_2 \in \mathbb{R}$$

$$\Rightarrow \boxed{a_4 = -(2k_1 + 3k_2)}$$

Put in (v)

$$[k_1, k_2 \in \mathbb{R}]$$

$$a_1 + k_1 + 2k_2 - 4k_1 - 6k_2 = 0$$

$$\Rightarrow \boxed{a_1 = 3k_1 + 4k_2}$$

Q Are these vectors $(1, 2, 1), (3, 1, 5), (3, -4, 7)$ are L.I or not

Solⁿ: Let $\alpha_1 = (1, 2, 1)$ and Let $a_1, a_2, a_3 \in \mathbb{F}$
 $\alpha_2 = (3, 1, 5)$ Such that
 $\alpha_3 = (3, -4, 7)$

$$a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 0$$

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$$\text{i.e. } a_1(1, 2, 1) + a_2(3, 1, 5) + a_3(3, -4, 7) = 0$$

On Comparing

$$a_1 + 3a_2 + 3a_3 = 0 \quad \text{--- (i)}$$

$$2a_1 + a_2 - 4a_3 = 0 \quad \text{--- (ii)}$$

$$a_1 + 5a_2 + 7a_3 = 0 \quad \text{--- (iii)}$$

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix} \quad \text{Coefficient matrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \mid R_3 \rightarrow R_3 - R_1$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 2 & 4 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{2}R_3 \mid R_2 \rightarrow \frac{1}{-5}R_2$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \rho(A) = 2 < (n=3)$$

Now

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

have

non zero Infinite Soln.

$$\begin{aligned} \text{Given infinite} \\ \text{Non zero soln} &= n - \rho(A) \\ &= 3 - 2 = 1 \end{aligned}$$

$$a_1 + 3a_2 + 3a_3 = 0 \quad \text{--- (iv)}$$

$$1a_2 + 2a_3 = 0 \quad \text{--- (v)}$$

$$\text{Let } a_3 = K$$

$$\Rightarrow a_2 = -2K$$

Put in (iv)

$$a_1 + 3(-2K) + 3K = 0$$

$$a_1 = 3K$$

$$\{K \in \mathbb{R}\}$$

Hence, Given Vectors are not L.I

Q Are the vectors $(2, 3, -1), (-1, 4, 2), (1, 18, 4)$ l.i or not.

Soln: Given $\alpha_1 = (2, 3, -1)$ let $a_1, a_2, a_3 \in F$ such that
 $\alpha_2 = (-1, 4, 2)$
 $\alpha_3 = (1, 18, 4)$
 $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$

$$\text{i.e. } a_1(2, 3, -1) + a_2(-1, 4, 2) + a_3(1, 18, 4) = 0$$

On comparing

$$2a_1 - a_2 + a_3 = 0$$

$$3a_1 + 4a_2 + 18a_3 = 0$$

$$-a_1 + 2a_2 + 4a_3 = 0$$

Coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 18 \\ -1 & 2 & 4 \end{bmatrix}$$

$$R_2 \rightarrow 2R_2 - 3R_1$$

$$R_3 \rightarrow 2R_3 + R_1$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 11 & 33 \\ 0 & 1 & 5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_1$$

$$R_2 = \frac{1}{11}R_2$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore \rho(A) = 3 = (n=3)$$

$\therefore$ It gives only one soln.

$$\text{i.e. } a_1 = a_2 = a_3 = 0 //$$

Hence, Given vector is linearly independent.

Imp

Q Prove that is vector space $V_n F$ system of n vectors
 $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $e_3 = (0, 0, 1, \dots, 0)$
 $\dots$ $e_n = (0, 0, 0, \dots, 1)$ is l.i where 1 denotes
the unit element $\in F$.

Soln: Given, $e_1 = (1, 0, 0, \dots, 0)$
 $e_2 = (0, 1, 0, \dots, 0)$
 $\dots$
 $e_n = (0, 0, 0, \dots, 1)$

Let $a_1, a_2, a_3, \dots, a_n \in F$
Such that

$$a_1e_1 + a_2e_2 + a_3e_3 + \dots + a_n e_n = 0$$

$$\text{i.e. } a_1(1,0,0,\dots,0) + a_2(0,1,0,\dots,0) + \dots + a_n(0,0,0,\dots,1) = 0$$

On Comparing

we get

$$a_1=0, a_2=0, a_3=0, \dots, a_n=0.$$

$\therefore$ Given vectors are Linearly Independent (L.I.)

Q Prove that set of vector $\{1, x, 1+x+x^2\}$ is L.I.

Set of vector space of all polynomials over the real number field.

Soln:- Let $a_1=1$

$$a_2=x$$

$$a_3=1+x+x^2$$

and Let $a_1, a_2, a_3 \in F$

Such that

$$a_1 a_1 + a_2 a_2 + a_3 a_3 = 0$$

$$\text{i.e. } a_1(1) + a_2(x) + a_3(1+x+x^2) = 0$$

$$\Rightarrow a_3 x^2 + (a_2 + a_3)x + (a_1 + a_3) = 0x^2 + 0x + 0$$

On Comparing

$$\boxed{a_3=0} \mid a_2+a_3=0 \mid a_1+a_3=0$$

$$\Rightarrow a_2+0=0 \Rightarrow a_2=0$$

$$\Rightarrow a_1+0=0 \Rightarrow a_1=0$$

$$\text{On Solving. } \Rightarrow \boxed{a_2=0} \Rightarrow \boxed{a_1=0}$$

$$\{a_1 \neq a_2 \neq a_3 = 0\}$$

Hence, Given set of vectors are L.I. proved //

Q If α, β, γ are L.I. vectors of $V(F)$ where F is field of Complex no. Then prove that

i) $\alpha+\beta, \alpha-\beta, \alpha-2\beta-\gamma$ are also L.I.

ii) $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also L.I.

Soln: i) Given, α, β, γ vectors are L.I.

To prove :-

Let $e_1=\alpha+\beta, e_2=\alpha-\beta, e_3=\alpha-2\beta-\gamma$ are L.I.

Let $a_1, a_2, a_3 \in F$

Such that

$$a_1 e_1 + a_2 e_2 + a_3 e_3 = 0$$

$$\text{i.e., } a_1(\alpha + \beta) + a_2(\alpha - \beta) + a_3(\alpha - 2\beta - \gamma) = 0$$

$$\Rightarrow a_1\alpha + a_1\beta + a_2\alpha - a_2\beta + a_3\alpha - 2a_3\beta - a_3\gamma = 0$$

$$\Rightarrow (a_1 + a_2 + a_3)\alpha + (a_1 - a_2 - 2a_3)\beta - a_3\gamma = 0$$

Given α, β, γ are L.I.

$$\therefore a_1 + a_2 + a_3 = 0 \quad \text{--- (i)}$$

$$a_1 - a_2 - 2a_3 = 0 \quad \text{--- (ii)}$$

$$a_3 = 0 \quad \text{--- (iii)}$$

Let coefficient matrix

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 1(-1-1) = -2 \neq 0$$

$$\Rightarrow |A| \neq 0$$

$$\therefore \rho(A) = 3 = n$$

System gives only one solⁿ.

$$\text{i.e. } a_1 = a_2 = a_3 = 0$$

$\therefore e_1, e_2, e_3$ are L.I.

ii) Given α, β, γ vectors of V_F is L.I.

To prove

$e_1 = \alpha + \beta$, $e_2 = \beta + \gamma$ and $e_3 = \gamma + \alpha$ are also L.I.

Let $a_1, a_2, a_3 \in F$

Such that

$$a_1 e_1 + a_2 e_2 + a_3 e_3 = 0$$

$$\text{ie, } a_1(\alpha + \beta) + a_2(\beta + \gamma) + a_3(\gamma + \alpha) = 0$$

$$\Rightarrow a_1\alpha + a_1\beta + a_2\beta + a_2\gamma + a_3\gamma + a_3\alpha = 0$$

$$\Rightarrow (a_1 + a_3)\alpha + (a_1 + a_2)\beta + (a_2 + a_3)\gamma = 0$$

Given α, β, γ are L.I

$$\therefore a_1 + a_3 = 0 \text{ --- (i)}$$

$$\text{On (i) + (ii) + (iii)}$$

$$a_1 + a_2 = 0 \text{ --- (ii)}$$

$$a_1 + a_2 + a_3 = 0 \text{ --- (iv)}$$

$$a_2 + a_3 = 0 \text{ --- (iii)}$$

$$\begin{array}{|l} \text{Put (i) in (iv)} \\ \boxed{a_2 = 0} \end{array} \quad \begin{array}{|l} \text{Put (ii) in (iv)} \\ \boxed{a_3 = 0} \end{array} \quad \begin{array}{|l} \text{Put (iii) in (iv)} \\ \boxed{a_1 = 0} \end{array}$$

$$\therefore a_1 = a_2 = a_3 = 0$$

$\therefore e_1, e_2, e_3$ are L.I //

Basis of Vector Space:-

Let $V(F)$ be a vector space over the field F .

Let $S \subseteq V(F)$, Then S is called a basis of $V(F)$.

i) S is L.I vectors

ii) Each vector in V is Linear Combination of finite no. of elements of S .

OR

Theorem:- Every linearly independent subset of finitely generated vector space $V(F)$ forms a part of basis of V .

Linear Transformation / Linear Mapping / Homomorphism of Vector Space

Defⁿ:- Let U_F and V_F be a two vector space over the field F .

A mapping $T: U \rightarrow V$ is called Linear transformation or linear mapping or vector space Homomorphism of U into V if T is such that

$$i) T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \alpha, \beta \in U$$

$$ii) T(a\alpha) = aT(\alpha) \quad \forall a \in U$$

To combine (i) and (ii), we get.

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \forall a, b \in F; \alpha, \beta \in U$$

Q If $T: V_3 \rightarrow V_2$ is defined as $T(x_1, x_2, x_3) = (x_2 - x_1, x_1 + x_3)$

Soln: Let $x = (x_1, x_2, x_3)$ $y = (y_1, y_2, y_3)$

$$\begin{aligned} T(x+y) &= T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \\ &= T[(x_1+y_1, x_2+y_2, x_3+y_3)] \\ &= (x_1+y_1 - x_2-y_2, (x_1+y_1) + x_3+y_3) \quad \left. \begin{array}{l} \text{--- by given} \\ \text{rule} \end{array} \right\} \\ &= [(x_1 - x_2) + (y_1 - y_2)], [(x_1 + x_3) + (y_1 + y_3)] \\ &= (x_1 - x_2, x_1 + x_3) + (y_1 - y_2, y_1 + y_3) \end{aligned}$$

$$T(x+y) = T(x) + T(y) \quad \dots \textcircled{I}$$

$$\begin{aligned} T(ax) &= T[a(x_1, x_2, x_3)] \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 - ax_2, ax_1 + ax_3) \\ &= a[(x_1 - x_2), (x_1 + x_3)] \end{aligned}$$

from ① & ②
it is clear that
the given mapping
is $L \cdot T$

$$T(ax) = a T(x_1, x_2, x_3) = a T(x) \quad \textcircled{II}$$

Q If $T: V_3 \rightarrow V_2$ by the rule $T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

Soln: Let $x = (x_1, x_2, x_3)$ $y = (y_1, y_2, y_3)$

$$\begin{aligned} T(x+y) &= T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \\ \Rightarrow T(x+y) &= T[(x_1+y_1, x_2+y_2, x_3+y_3)] \\ &= (x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 \quad \dots \textcircled{I} \end{aligned}$$

Also

$$\begin{aligned} T(x) + T(y) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \\ T(x) + T(y) &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + (x_3^2 + y_3^2) \quad \dots \textcircled{II} \end{aligned}$$

from (i) and (ii)

$$T(x+y) \neq T(x) + T(y)$$

∴ The given relation is not linear transformation.

Q. If $T: V_3 \rightarrow V_2$ is defined as $T(x, y, z) = (y, x)$

Then show that T is linear

Soln: Let $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$

$$\begin{aligned} T(x+y) &= T[(x_1, y_1, z_1) + (x_2, y_2, z_2)] \\ &= T[(x_1+x_2), (y_1+y_2), (z_1+z_2)] \\ &= [(y_1+y_2), (x_1+x_2)] \\ &= (y_1, x_1) + (y_2, x_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ T(x+y) &= T(x) + T(y) \quad \text{--- (i)} \end{aligned}$$

Also,

$$\begin{aligned} T(ax) &= T[a(x_1, y_1, z_1)] \\ &= T[ax_1, ay_1, az_1] \\ &= (ay_1, ax_1) \\ &= a(y_1, x_1) \\ &= aT(x_1, y_1, z_1) \end{aligned}$$

$$T(ax) = aT(x) \quad \text{--- (ii)}$$

Hence, T is linear transformation (L.T)

Q Show that the mapping $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by

$$T(a, b, c) = (c, a+b) \text{ is L.T.}$$

Solⁿ: Let $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$

Now

$$\begin{aligned} T(A+B) &= T[(a_1, b_1, c_1) + (a_2, b_2, c_2)] \\ &= T[(a_1+a_2), (b_1+b_2), (c_1+c_2)] \\ &= (c_1+c_2), (a_1+a_2+b_1+b_2) \\ &= (c_1+c_2), (a_1+b_1+a_2+b_2) \\ &= \{c_1, (a_1+b_1)\}, \{c_2, (a_2+b_2)\} \\ &= T(a_1, b_1, c_1) + T(a_2, b_2, c_2) \end{aligned}$$

$$T(A+B) = T_A + T_B \quad \dots \quad (i)$$

$$\begin{aligned} T(aA) &= T(aa_1, ab_1, ac_1) \\ &= [ac_1, (aa_1+ab_1)] \\ &= a[c_1, (a_1+b_1)] \\ &= aT(a_1, b_1, c_1) \end{aligned}$$

$$\Rightarrow T(aA) = aT_A \quad \dots \quad (ii)$$

from (i) & (ii)

Given mapping is L.T.

Q. $T: V_2 \rightarrow V_3$; defined by $T(x, y) = (x^3, y^3)$ is L.T or not.

Soln: Let $X = (x_1, y_1)$ & $Y = (x_2, y_2)$

Now

$$\begin{aligned} \bullet T_{(X+Y)} &= T[(x_1, y_1) + (x_2, y_2)] \\ &= T[(x_1+x_2), (y_1+y_2)] \quad \left\{ \begin{array}{l} \text{By defn} \\ \dots \textcircled{i} \end{array} \right. \\ \Rightarrow T_{(X+Y)} &= [(x_1+x_2)^3, (y_1+y_2)^3] \end{aligned}$$

$$\begin{aligned} \bullet T_X + T_Y &= T_{(x_1, y_1)} + T_{(x_2, y_2)} \\ &= (x_1^3, y_1^3) + (x_2^3, y_2^3) \end{aligned}$$

$$T_X + T_Y = [(x_1^3 + x_2^3), (y_1^3 + y_2^3)] \quad \dots \textcircled{ii}$$

from $\textcircled{i}$ and $\textcircled{ii}$

It is clear that

$$T_{(X+Y)} \neq T_X + T_Y$$

Hence, Given mapping is not L.T.

Q. $T: V_2 \rightarrow V_3$: Defined by $T(x, y) = \{2x+3y, (3x-4y)\}$. Show that T is L.T.

Soln Let $X = (x_1, y_1)$ and $Y = (x_2, y_2)$

Now

$$\begin{aligned} T_{(X+Y)} &= T[(x_1, y_1) + (x_2, y_2)] \\ &= T[(x_1+x_2), (y_1+y_2)] \\ &= [\{2(x_1+x_2) + 3(y_1+y_2)\}, \{3(x_1+x_2) - 4(y_1+y_2)\}] \\ &= [\{2x_1+2x_2+3y_1+3y_2\}, \{3x_1+3x_2-4y_1-4y_2\}] \end{aligned}$$

$$\Rightarrow T(x+y) = [\{(2x_1+3y_1)+(2x_2+3y_2)\}, \{(3x_1-4y_1)+(3x_2-4y_2)\}]$$

$$= [\{(2x_1+3y_1), (3x_1-4y_1)\} + \{(2x_2+3y_2), (3x_2-4y_2)\}]$$

$$\Rightarrow T(x+y) = T(x) + T(y) \quad \dots \quad (i)$$

Also,

$$\bullet T(ax) = T[a\{(2x_1+3y_1), (3x_1-4y_1)\}] = T(ax_1+ay_1)$$

$$\Rightarrow T(ax) = a T(x) \quad \dots \quad (ii)$$

from (i) & (ii)

It is clear that, given mapping is LT.

Some Theorem of Vector space:-

Theorem 01:- The necessary and sufficient condition for a non empty subset "W" of a vector space "V" to be a vector subspace of V is

$$a, b \in F ; \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W.$$

Proof:- (i) Condition is necessary:-

Let W be a subspace of V. Then we shall prove that

$$a, b \in F ; \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$$

Now,

let $a \in F, \alpha \in W \Rightarrow a\alpha \in W$ $\left\{ \begin{array}{l} W \text{ is subspace of } V \\ \therefore \text{ It is closed w.r.t.} \\ \text{Scalar multiplication} \end{array} \right\}$

$$\therefore a\alpha \in W ; b\beta \in W \Rightarrow a\alpha + b\beta \in W$$

[W holds the closure law for addition]

II. Condition is sufficient :-

Let W be a non empty subset of V_F and $a\alpha \in W$,
 $b\beta \in W$

$$\Rightarrow a\alpha + b\beta \in W \dots \textcircled{1}$$

Then, we shall prove that W is subspace of V_F

Now

i) Put $a=1, b=1$ in $\textcircled{1}$

$$\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$$

$\therefore W$ is closed w.r.t addition $\oplus$

ii) let $a=-1, b=0, \in F$ in $\textcircled{1}$

$$-\alpha \in W, 0\beta \in W \Rightarrow -\alpha \in W$$

$\therefore$ Additive Inverse exist.

iii) Let $a=0, b=0$ in $\textcircled{1}$

$$0\alpha \in W, 0\beta = W \Rightarrow \alpha + 0\beta \in W \\ \Rightarrow 0 \in W$$

$\therefore$ Additive Identity exist.

iv) Put $b=0$ in $\textcircled{1}$

$$a\alpha \in W \Rightarrow a\alpha + 0\beta \in W \\ \Rightarrow a\alpha \in W$$

Hence, $W \subseteq F$

Consequently the vector addition is commutative as well as associative in W . Also the remaining conditions of vector space hold in W .

So, finally W is subspace of V_F

Theorem 02:- The intersection of any two subspaces of vector space V_F is also a subspace of V_F
OR

If W_1 and W_2 be two subspaces of V_F
Then, Prove that $W_1 \cap W_2$ is subspace of V_F .

Proof:- Let W_1 and W_2 be a two subspaces of V_F .

Now,

$$0 \in V \Rightarrow 0 \in W_1 \text{ and } 0 \in W_2 \\ \Rightarrow 0 \in W_1 \cap W_2$$

$$\text{Let } \alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1, \alpha \in W_2$$

$$\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1, \beta \in W_2$$

$$\text{Let } a, b \in F; \alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1 \text{ [By Theorem 1]}$$

$$a, b \in F; \alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$$

Therefore,

$$a\alpha + b\beta \in W_1 \cap W_2$$

Hence, $W_1 \cap W_2$ is subspace of V_F

proved

Problem 1: Let W be the set of the form $(x, 2x, -3x, x)$,
Then Prove that W is a subspace of V_F .

Soln

$$\text{let } \alpha = (x, 2x, -3x, x); \beta = (y, 2y, -3y, y)$$

let a & $b \in F$ then,

$$a\alpha + b\beta = a(x, 2x, -3x, x) + b(y, 2y, -3y, y)$$

$$\Rightarrow a\alpha + b\beta = [(ax+by), (2ax+2by), (3ax-3by), (ax+by)] \\ \in W$$

$\therefore a\alpha + b\beta \in W$ because $ax+by \in F$

$\therefore W$ is subspace of V_F .

Problem 2 If $W = \{(a_1, a_2, a_3) : a_1, a_3 \in F \text{ \& } a_1 + a_2 + a_3 = 0\}$, Prove that W is subspace of $V_3 F$.

Solⁿ: let $\alpha = (a_1, a_2, a_3) \mid \beta = (b_1, b_2, b_3) \in W$. . . (i)

let $a_1, a_2, a_3 \in F$; $b_1, b_2, b_3 \in F$ Such that
 $a_1 + a_2 + a_3 = 0 \mid b_1 + b_2 + b_3 = 0$

Now,
$$\alpha\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3)$$

$$= [(aa_1 + bb_1), (aa_2 + bb_2), (aa_3 + bb_3)] \text{ . . . (ii)}$$

Now,
$$aa_1 + bb_1 + aa_2 + bb_2 + aa_3 + bb_3$$

$$= a(a_1 + a_2 + a_3) + b(b_1 + b_2 + b_3) = a \cdot 0 + b \cdot 0 = 0$$

from (ii) ; It is clear

$$\alpha\alpha + b\beta = \{(aa_1 + bb_1), (aa_2 + bb_2), (aa_3 + bb_3)\}$$

$\therefore W$ is subspace of V_F .

Problem 3 Prove that necessary and sufficient condⁿ for a non empty set W of vector space V_F to be a subspace of V is that W is closed under the space multiplication & scalar addition.

Solⁿ Proof: (I) Condⁿ is necessary:- let W be a subspace of V_F , Then we shall to prove that

$$\alpha, \beta \in W \Rightarrow \alpha + \beta \in W \mid a \in F, \alpha \in W \Rightarrow a\alpha \in W$$

By defiⁿ of vector^{sub}space of W , itself is a vector space over F wrt vector addition and scalar multiplication.

So (i) and (ii) holds.

(II) Condⁿ is sufficient:- let $W \subseteq V_F$ & given $\alpha, \beta \in W$
 $a \in F, \alpha \in W \Rightarrow a\alpha \in W \Rightarrow \alpha + \beta \in W$

Now We have to prove that, W is subspace of V_F .

i) $\alpha \in W$; $1 \in F \Rightarrow \alpha \in W$, $-1 \in F$ (Additive Inverse)

$\therefore \alpha \in W, (-1) \in F \Rightarrow -\alpha \in W$.

ii) $\alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W$ { W is closed wrt addition }

$\therefore 0 \in W$ { Additive identity exist } / Since, $W \subseteq F$ //

Consequently, the vector addition is commutative as well as associative for W . Also the remaining condⁿ of a vector space hold in W .
 So, finally W is subspace of V_F .

MATRIX

Topic to be done

- i) Eigen value and Eigen vector
- ii) Cayley Hamilton Theorem
- iii) Rank of matrix
- iv) Solⁿ of simultaneous equations

I. Eigen Value and Eigen Vector

1) Eigen Values:-

Let 'A' be an any square matrix and λ be any scalar. Then $|A - \lambda I| = 0$ is called a characteristic equation or a Eigen equation. And

$\lambda = \lambda_1, \lambda_2, \lambda_3, \dots$ are called Eigen values.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ find eigen value and eigen vector

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Characteristic eqⁿ $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\text{i.e. } 1-\lambda = 0, \lambda = 1$$

$$2-\lambda = 0, \lambda = 2$$

$$3-\lambda = 0, \lambda = 3$$

$\therefore$ Eigen values are 1, 2, 3.

To find eigen vector's for which

$$[A - \lambda I]X = 0$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 0x_1 + 2x_2 + 3x_3 = 0 \Rightarrow 2x_2 + 3x_3 = 0 \quad \text{--- (i)}$$

$$0x_1 + x_2 + 4x_3 = 0 \Rightarrow x_2 + 4x_3 = 0 \quad \text{--- (ii)}$$

$$0x_1 + 0x_2 + 3x_3 = 0 \Rightarrow 3x_3 = 0 \quad \text{--- (iii)}$$

From (iii) $x_3 = 0$ | From (i) $x_2 = 0$ $x_1 = K(1, 2, 3, \dots)$

$$\therefore E_1 = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 2$

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 + 3x_3 = 0 \quad \text{--- (i)}$$

$$4x_3 = 0$$

$$\Rightarrow x_3 = 0 \quad \text{--- (ii) put in (i) } \Rightarrow x_1 = 2x_2$$

$$-x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = K \text{ (say)} \neq 0$$

$$E_2 = \begin{bmatrix} 2K \\ K \\ 0 \end{bmatrix} = K \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Put $\lambda = 3$

$$\begin{bmatrix} -2 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \quad \text{--- (1)}$$

$$-x_2 + 4x_3 = 0$$

$$x_2 = 4x_3$$

$$\frac{x_2}{4} = \frac{x_3}{1} = K \text{ (say)} \Rightarrow x_2 = 4K, x_3 = K$$

Put in (1)

$$x_1 = \frac{11}{2}K$$

$$E_3 = \begin{bmatrix} \frac{11}{2}K \\ 4K \\ K \end{bmatrix} = \frac{K}{2} \begin{bmatrix} 11 \\ 8 \\ 2 \end{bmatrix}$$

NOTE:-

- i) Eigen vector are in non-zero form.
- ii) Any variable (i.e x_1, x_2, x_3) does not present in any equation
So say it's $K (K \neq 0)$
- iii) Eigen vector are in ration form.
- iv) more Eigen vector corresponding to one Eigen values.
- v) If two value of λ are equal, so we get always a single equation x_1, x_2 & x_3 .
- vi) Formula:- Take any two equation.

$$\frac{x_1}{p-0} = \frac{x_2}{p-0} = \frac{x_3}{p-0}$$

$$\underline{Q} \quad A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

Solⁿ: Characteristic Equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - (-5)(-2) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda+1)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -1 \text{ and } 6$$

Put $\lambda = -1$

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0 \quad \text{--- (i)}$$

$$-5x_1 + 5x_2 = 0 \quad \text{--- (ii)}$$

$$x_1 = x_2 = K \text{ (say)}$$

$$E_1 = \begin{bmatrix} K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Put $\lambda = 6$

$$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-5x_1 - 2x_2 = 0 \quad \text{--- (i)}$$

$$-5x_1 - 2x_2 = 0 \quad \text{--- (ii)}$$

$$\frac{x_1}{2} = \frac{x_2}{-5} = K \text{ (say)}$$

$$x_1 = 2K, x_2 = -5K$$

$$E_2 = \begin{bmatrix} 2K \\ -5K \end{bmatrix} = K \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$Q \quad A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln: Characteristic equation

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(1-\lambda)(3-\lambda)+4] + 6[(3-\lambda)(-6)+8] + 2[24-14+\lambda] = 0$$

$$\Rightarrow (8-\lambda)[\lambda^2 - 10\lambda + 5] + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow -\lambda(\lambda-3)(\lambda-15) = 0$$

$$\therefore \lambda = 0, 3, 15$$

For Eigen vector

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \dots \textcircled{1}$$

Put $\lambda = 0$ in $\textcircled{1}$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \quad \dots \textcircled{i}$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \quad \dots \textcircled{ii}$$

$$2x_1 - 4x_2 + 3x_3 = 0 \quad \dots \textcircled{iii}$$

From $\textcircled{ii}$ & $\textcircled{iii}$

$$\frac{x_1}{21-18} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14}$$

$$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

$$\therefore x_1 = \frac{x_2}{2} = \frac{x_3}{2} = k \text{ (say)}$$

$$E_1 = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Put $\lambda = 3$ in (i)

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \quad \text{--- (i)}$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \quad \text{--- (ii)}$$

$$2x_1 - 4x_2 = 0 \quad \text{--- (iii)}$$

from (iii)

$$2x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{2} = k$$

$$\therefore x_1 = 4k, x_2 = 2k$$

Put in (i)

$$20k - 12k + 2x_3 = 0$$

$$2x_3 = -8k$$

$$\Rightarrow x_3 = -4k$$

$$\therefore E_2 = \begin{bmatrix} 4k \\ 2k \\ -4k \end{bmatrix}$$

$$= k \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$

Put $\lambda = 15$ in (i)

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \quad \text{--- (i)}$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \quad \text{--- (ii)}$$

$$2x_1 - 4x_2 - 12x_3 = 0 \quad \text{--- (iii)}$$

from (i) and (iii)

$$\frac{x_1}{24+16} = -\frac{x_2}{28+12} = \frac{x_3}{56-36}$$

$$\Rightarrow \frac{x_1}{40} = -\frac{x_2}{40} = \frac{x_3}{20}$$

$$\Rightarrow \frac{x_1}{2} = -\frac{x_2}{2} = \frac{x_3}{1} = k \text{ (say)}$$

$$\therefore E_3 = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} //$$

$$Q: A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$$

Soln. char. eqn

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)[(-2-\lambda)(3-\lambda) + 4] + 4[(3-\lambda) - 4] + 4[-1 - (-2-\lambda)] = 0$$

$$\Rightarrow (3-\lambda)[-6 + 2\lambda - 3\lambda + \lambda^2 + 4] + 4(-\lambda - 1) + 4(\lambda + 1) = 0$$

$$\Rightarrow (3-\lambda)(\lambda + 1)(\lambda - 2) - 4(\lambda + 1) + 4(\lambda + 1) = 0$$

$$\Rightarrow (3-\lambda)(\lambda + 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 3, -1, 2$$

For Eigen vector

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (I)}$$

Put $\lambda = -1$ in eq (I)

$$\begin{bmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 4x_2 + 4x_3 = 0 \quad \text{--- (I)}$$

$$x_1 - x_2 + 4x_3 = 0 \quad \text{--- (II)}$$

$$x_1 - x_2 + 4x_3 = 0 \quad \text{--- (III)}$$

from (I) & (II)

$$\frac{x_1}{-16 + 4} = \frac{-x_2}{16 - 4} = \frac{x_3}{-4 + 4} = k \text{ (say)}$$

$$\therefore x_1 = -12k \quad | \quad x_2 = 12k \quad | \quad x_3 = 0$$

$$\therefore E_1 = \begin{bmatrix} -12k \\ 12k \\ 0 \end{bmatrix} = K' \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{K' = 12k}$ say

Put $\lambda = 2$ in (i)

$$\begin{bmatrix} 1 & -4 & 4 \\ 1 & -4 & 4 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 4x_2 + 4x_3 = 0 \quad \text{--- (i)}$$

$$x_1 + 4x_3 = 0 + 4x_2 \quad \text{--- (ii)}$$

$$x_1 - x_2 + x_3 = 0 \quad \text{--- (iii)}$$

from (ii) & (iii)

$$x_2 = x_3 \mid x_1 = 0 \mid x_3 = x_3 = k (\text{say})$$

$$\therefore \mathbf{E}_2 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} //$$

Put $\lambda = 3$ in (i)

$$\begin{bmatrix} 0 & -4 & 4 \\ 1 & -5 & 4 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_2 + 4x_3 = 0 \quad \text{--- (i)}$$

$$1x_1 - 5x_2 + 4x_3 = 0 \quad \text{--- (ii)}$$

$$1x_1 - x_2 + 0 = 0 \quad \text{--- (iii)}$$

from (i) $x_3 = x_2 = k (\text{say})$

Put in (ii)

$$x_1 = +5k - 4k = k$$

$$\therefore \mathbf{E}_3 = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} //$$

$$iv) A = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

Solⁿ. Characteristic eqⁿ $|A - \lambda I| = 0$

$$\begin{vmatrix} \cos \theta - \lambda & \sin \theta \\ \sin \theta & -\cos \theta - \lambda \end{vmatrix} = 0$$

$$-(\cos \theta - \lambda)(\cos \theta + \lambda) - \sin^2 \theta = 0$$

$$\Rightarrow -\cos^2 \theta + \lambda^2 - \sin^2 \theta = 0$$

$$\Rightarrow -1 + \lambda^2 = 0$$

$$\Rightarrow \lambda = \pm 1$$

for Eigen vector

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} \cos \theta - 1 & \sin \theta \\ \sin \theta & -(\cos \theta - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots \textcircled{1}$$

Put $\lambda = 1$

$$\begin{bmatrix} \cos \theta - 1 & \sin \theta \\ \sin \theta & -(\cos \theta + 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1(\cos \theta + 1) - \sin \theta x_2 = 0 \quad \dots \textcircled{I}$$

$$\sin \theta x_1 - (\cos \theta + 1)x_2 = 0 \quad \dots \textcircled{II}$$

from $\textcircled{I}$

$$\frac{x_1}{\cos \theta - 1} = \frac{x_2}{\sin \theta} = K \text{ (say)}$$

$$\therefore E_1 = K \begin{bmatrix} \cos \theta - 1 \\ \sin \theta \end{bmatrix}$$

Put $\lambda = -1$

$$\begin{bmatrix} \cos \theta + 1 & \sin \theta \\ \sin \theta & 1 - \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\cos \theta + 1)x_1 + (\sin \theta)x_2 = 0 \quad \dots \textcircled{I}$$

$$\sin \theta x_1 + (1 - \cos \theta)x_2 = 0 \quad \dots \textcircled{II}$$

from $\textcircled{II}$

$$\frac{x_1}{\cos \theta - 1} = \frac{x_2}{\sin \theta} = K \text{ (say)}$$

$$\therefore E_2 = K \begin{bmatrix} \cos \theta - 1 \\ \sin \theta \end{bmatrix}$$

V)

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Characteristic eqⁿ

$$|A - \pi I| = 0$$

$$\begin{vmatrix} 2-\pi & 2 & 1 \\ 1 & 3-\pi & 1 \\ 1 & 2 & 2-\pi \end{vmatrix} = 0$$

$$\Rightarrow R_1 \rightarrow R_1 - R_2 \mid R_2 \rightarrow R_2 - R_3$$

$$\begin{vmatrix} 1-\pi & -1+\pi & 0 \\ 0 & 1-\pi & -1+\pi \\ 1 & 2 & 2-\pi \end{vmatrix} = 0$$

$$\Rightarrow \left[\begin{array}{l} \text{Expand along } C_1 \\ (1-\pi)[(2-\pi)(2-\pi)] + 1[(\pi+1)^2 - 0] = 0 \\ (1-\pi)[(1-\pi)(2-\pi) - (2-\pi)] + [(\pi+1)^2 - 0] = 0 \\ -(1-\pi)[(2-\pi) + 2 + (1-\pi)] \end{array} \right]$$

$$= (\pi-1)(\pi-1) \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & 2-\pi \end{vmatrix} = 0$$

$$\emptyset \quad C_2 \rightarrow C_2 + C_1$$

$$= (\pi-1)^2 \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 3 & 2-\pi \end{vmatrix} = 0$$

$$= (\pi-1)^2 \times -1(-2+\pi-3)$$

$$= (\pi-1)(\pi-1)(5-\pi) = 0$$

$$\Rightarrow \pi = 1, 4, 5$$

For Eigen vector.

$$[A - \pi I]X = 0$$

$$\begin{bmatrix} 2-\pi & 2 & 1 \\ 1 & 3-\pi & 1 \\ 1 & 2 & 2-\pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Put $\pi = 1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0$$

i) let $x_1 = 2, x_2 = 1$

$$\therefore 2 + 2 + x_3 = 0 \\ \Rightarrow x_3 = -4$$

$$\therefore E_1 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

ii) let $x_2 = 0, x_1 = 5$

$$\therefore 5 + 0 + x_3 = 0 \\ x_3 = -5$$

$$\therefore E_2 = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

Put $\pi = 5$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0 \quad \text{--- (i)}$$

$$x_1 - 2x_2 + x_3 = 0 \quad \text{--- (ii)}$$

$$x_1 + 2x_2 + 3x_3 = 0 \quad \text{--- (iii)}$$

from (i) & (ii)

$$\frac{x_1}{6-2} = \frac{-x_2}{-3-1} = \frac{x_3}{2+2} = k(\text{say})$$

$$\text{i.e. } \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

$$\therefore E_3 = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_1 \rangle A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solⁿ Characteristic eqⁿ

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda) \{ (3-\lambda)^2 - 1 \} - 2 \{ (-2)(3-\lambda) + 2 \} + 2 \{ 2 - 6 + 2\lambda \} = 0$$

$$\Rightarrow (6-\lambda) \{ 9 - 6\lambda + \lambda^2 - 1 \} - 2 \{ -6 + 2\lambda + 2 \} + 2 \{ 2 - 6 + 2\lambda \} = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 8)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 2, 2, 8$$

for Eigen vector

$$[A - \lambda I]x \neq 0$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

For $\lambda = 2$ eq (1)

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} 4x_1 - 2x_2 + 2x_3 = 0 \quad \text{--- (i)} \\ -2x_1 + x_2 - x_3 = 0 \quad \text{--- (ii)} \\ 2x_1 - x_2 + x_3 = 0 \quad \text{--- (iii)} \end{array} \right\} \rightarrow \underline{2x_1 - x_2 + x_3 = 0}$$

Let $x_1 = 1, x_2 = 0 \Rightarrow x_3 = -2$

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\text{let } x_1 = 2, x_2 = 1 \Rightarrow x_3 = 5$$

$$E_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Put $\pi = 8$ in (i)

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{--- (i)}$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{--- (ii)}$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \text{--- (iii)}$$

on (i) + (iii)

$$-3x_2 - 3x_3 = 0$$

$$\Rightarrow x_2 = -x_3 = k \text{ (say)}$$

Put in (i)

$$-2x_1 = 2k + 2k$$

$$\Rightarrow x_1 = -2k$$

$$E_3 = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

VII) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Soln: Char. eqn $|A - \pi I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\pi & 1 & 1 \\ 1 & 1-\pi & 1 \\ 1 & 1 & 1-\pi \end{vmatrix} = 0$$

$\otimes R_1 \rightarrow R_1 + R_2 + R_3$

$$\Rightarrow (3-\pi) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\pi & 1 \\ 1 & 1 & 1-\pi \end{vmatrix} = 0$$

$\otimes R_1 \rightarrow R_1 - R_2 \mid R_2 \rightarrow R_2 - R_3$

$$\Rightarrow (3-\pi) \begin{vmatrix} 0 & \pi & 0 \\ 0 & -\pi & \pi \\ 1 & 1 & 1-\pi \end{vmatrix} = 0$$

$$\Rightarrow -(3-\pi)\pi^2 \begin{vmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1-\pi \end{vmatrix} = 0$$

$$\Rightarrow \pi^2(3-\pi) \cdot 1 \cdot [-1] = 0$$

$$\Rightarrow \pi = 0, \pi = 3 //$$

for Eigen vector

$$[A - \pi I]X = 0$$

$$\begin{bmatrix} 1-\pi & 1 & 1 \\ 1 & 1-\pi & 1 \\ 1 & 1 & 1-\pi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{--- (1)}$$

(Put $\pi = 0$) in (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$

i) let $x_1 = 2, x_3 = 2$
 $\Rightarrow x_2 = -4$

$$E_1 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$$

ii) let $x_1 = 0, x_2 = -2$
 $\Rightarrow x_3 = 2$

$$E_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

(Put $\pi = 3$) in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + x_3 = 0 \quad \dots \textcircled{i}$$

$$x_1 - 2x_2 + x_3 = 0 \quad \dots \textcircled{ii}$$

$$x_1 + x_2 - 2x_3 = 0 \quad \dots \textcircled{iii}$$

On $\textcircled{i} + 2\textcircled{ii}$

$$0 - 3x_2 + 3x_3 = 0$$

$$\Rightarrow x_2 = x_3 = k \text{ (say)}$$

Put in $\textcircled{ii}$

$$x_1 = k$$

$$E_3 = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} //$$

**** Easy N Imp ***

Cayley Hamilton Theorem :-

Every square matrix 'A' satisfy its characteristic equation.

$$\text{i.e. } |A - \lambda I| = 0$$

Ex. $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \dots \textcircled{i}$

According to Cayley Hamilton Theorem verify that

~~$$A^3 - 6A^2 + 9A - 4I = 0$$~~

{Put $\lambda = A$ in eqⁿ $\textcircled{i}$ }

Verify Cayley Hamilton Theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence evaluate A^{-1} .

Solⁿ:- Ch. eqⁿ:- $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

~~$$\Rightarrow (2-\lambda) \{ (2-\lambda)^2 - (1) \} + 1 \{ (-2+\lambda) - (-1) \} + 1 \{ 1 - 2 + \lambda \} = 0$$~~

$$\Rightarrow (2-\lambda)\{\lambda^2-4\lambda+3\} + \{\lambda+1\} + \{\lambda+1\} = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \text{--- (1)}$$

→ By Cayley Hamilton theorem

Let characteristic equation

$$\lambda^3 - P\lambda^2 + Q\lambda - |A|I = 0$$

Where, • P = Sum of Principal diagonal elements

$$\Rightarrow P = 6 = 2+2+2$$

• Q = Sum of the minor of diagonal elements

$$\Rightarrow Q = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 9$$

Also

$$|A| = 2(4-1) - (-1)(-2+1) + 1(1-2) = 4$$

∴ Ch. eqn

$$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4I = 0 //$$

→ To verify Cayley Hamilton theorem for which Put $\lambda = A$ in (1)

$$\text{i.e.: } A^3 - 6A^2 + 9A - 4I = 0 \quad \text{--- (2)}$$

Now

$$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+4+4 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Also,

$$A^3 = AA^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$$\underline{LHS} = A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21-(-30)-9 & 21-30+9 \\ -21-(-30)-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & 21+30-9 & 22+36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3 \times 3} \quad \text{Hence proved} \\ = \underline{RHS}$$

Now

from (2)

multiply by A^{-1}

$$\Rightarrow A^2 - 6A + 9AA^{-1} - 4IA^{-1} = 0A^{-1}$$

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4} [A^2 - 6A + 9I]$$

$$\Rightarrow A^{-1} = \frac{1}{4} \left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right]$$

$$\Rightarrow A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} //$$

Q Verify Cayley Hamilton theorem for the following matrix A.

$$i) A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix}$$

Soln: let Ch. eqn: $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - P\lambda^2 + Q\lambda - |A|I = 0 \quad \text{--- (1)}$$

- $P = \text{sum of diag. element}$
 $= 1 + 3 - 4 = 0$

- $Q = \text{sum of minor of diag. element}$

$$= \begin{vmatrix} 3 & 3 \\ -4 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix}$$

$$= 0 + \{-4 - (-6)\} + \{+3 - (-4)\}$$

$$= 2(2) = 4$$

Also,

$$|A| = 1(0) + 1(2) + 3(2) = 4$$

$$\therefore \text{Ch eqn is } \lambda^3 + 4\lambda + 4I = 0 \quad \text{--- (2)}$$

By Cayley Hamilton theorem.

$$A^3 + 4A + 4I = 0 \quad \text{--- (2)}$$

$$AA = A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1+1-6 & 1+3-12 & 3+3-12 \\ 1+9-6 & 1+9-12 & 3+9-12 \\ -2-4+8 & -2-12+16 & -6-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -6 \\ -2 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix}$$

Also,

$$AA^2 = A^3 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} -4 & -8 & -6 \\ -2 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} +2 & +4 & +3 \\ +1 & 1 & 0 \\ -1 & -1 & +1 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 2+1-3 & 4+1-3 & +3+0+3 \\ 2+3-3 & 4+3-3 & +3+0+3 \\ -4-4+4 & -8-4+4 & +6+0+4 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 0 & 2 & 6 \\ 2 & 4 & 6 \\ -4 & -8 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 & -12 \\ -4 & -8 & -12 \\ 8 & 16 & 10 \end{bmatrix}$$

Now, from (1)

$$\begin{bmatrix} 0 & -4 & -12 \\ -4 & -8 & -12 \\ 8 & 16 & 10 \end{bmatrix} + 4 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 3 \\ -2 & -4 & -4 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-4+4 & -4+4+0 & -12+12+0 \\ -4+4+0 & -8+12-4 & -12+12-0 \\ 8-8+0 & 16-16-4 & 20-16-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiply by A^{-1} in eq (2)

$$A^3 A^{-1} + 4 A^{-1} A - 4 I A^{-1} = 0 A^{-1}$$

$$\Rightarrow A^2 + 4I - 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = -\frac{1}{4} [-A^2 - 4I]$$

$$\Rightarrow A^{-1} = -\frac{1}{4} \left\{ \begin{bmatrix} -4 & -8 & -6 \\ -2 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix} \right\} - \frac{4}{8} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -\frac{1}{4} \left\{ \begin{bmatrix} 0 & 8 & 6 \\ 2 & -2 & 0 \\ -2 & -2 & -2 \end{bmatrix} \right\}$$

$$\Rightarrow A^{-1} = \frac{1}{4} \begin{bmatrix} 0 & -8 & -6 \\ -2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -4 & -3 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} //$$

$$ii) A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Soln? let char. eqn: $|A + \lambda I| = 0$

$$\Rightarrow \lambda^3 - P\lambda^2 + Q\lambda - |A|I = 0 \quad \text{--- (1)}$$

$$\bullet P = 1 + 2 + 1 = 4$$

$$\bullet Q = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= 2 + 6 + 1 - 7 + 2 - 12$$

$$= 5 - 25 = -20$$

Also

$$|A| = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} + 7 \begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 2 - 6 - 3 + 7 \times 6$$

$$= 35$$

$\therefore$ Char. eqn is

$$\lambda^3 - 4\lambda^2 - 20\lambda + 35I = 0$$

By Cayley Hamilton theorem

$$A^3 - 4A^2 - 20A + 35I = 0 \quad \text{--- (2)}$$

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

Now

$$A^3 = A^2 A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

from ②

$$LHS = A^3 - 4A^2 - 20A - 35I = 0 = RHS$$

$$\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now

multiply by A^{-1} in eq ②

$$A^3 A^{-1} - 4A^2 A^{-1} - 20A A^{-1} - 35 A^{-1} = 0 A^{-1}$$

$$\Rightarrow A^2 - 4A - 20I - 35 A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{35} \{ A^2 - 4A - 20I \}$$

$$\Rightarrow A^{-1} = \frac{1}{35} \left\{ \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \right\}$$

$$\Rightarrow A^{-1} = \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} //$$

$$\text{iii)} \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

Soln Char. eqn: $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - p\lambda^2 + q\lambda - |A|I = 0 \quad \text{--- (1)}$$

$$p = 1+2+3 = 6$$

$$q = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 6 + 3 - 4 + 2 = 7$$

Also

$$|A| = 1 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} - 0(-) + 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 6 + (-8) = -2$$

$$\therefore \text{Char. eqn is } \lambda^3 - 6\lambda^2 + 7\lambda + 2I = 0$$

By Cayley Hamilton theorem

$$A^3 - 6A^2 + 7A + 2I = 0 \quad \text{--- (2)}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

Now

$$A^2 A = A^3 = \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 0 & 38 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

From (2)

$$LHS = A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 23 & 0 & 38 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} //$$

Now multiply by A^{-1} in eq (2)

$$A^2 - 6A + 7I + 2A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{2}[-A^2 + 6A - 7I]$$

$$\Rightarrow A^{-1} = \frac{1}{2} \left\{ - \begin{bmatrix} 7 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right\}$$

$$= \frac{1}{2} \left\{ \begin{bmatrix} -8 & 0 & 4 \\ 2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix} \right\}$$

$$\Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} -8 & 0 & 4 \\ 2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix} //$$

Rank of Matrix :-

{Notation $\rho(A)$ or r }

→ Methods to find Rank of matrix

i) Direct method (minor method)

ii) Reduction in normal form.

iii) Echelon form.

(I) Direct method (minor method):-

i) At least one minor of order 3 is not zero

$$\text{i.e.; } |A|_{3 \times 3} \neq 0$$

$$\therefore \rho(A) = 3$$

→ If all minor of order 3 are zero then $\rho(A) < 3$

ii) At least one minor of order 2 is known zero.

$$\text{i.e. } |A|_{2 \times 2} \neq 0$$

$$\therefore \rho(A) = 2$$

→ If all minor of order 2 are zero then $\rho(A) < 2$

iii) At least one minor of order one (1).

{each element each self a minor} are not zero.

$$\therefore \rho(A) = 1$$

Q Find rank of matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 2 & 4 & -3 \end{bmatrix}$

step 1 At least one minor of $3 \times 3 \neq 0$

Soln

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{vmatrix} = 0$$

$$\therefore \rho(A) < 3$$

step 2 At least one minor of $2 \times 2 \neq 0$

$$|A|_{2 \times 2} = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 \neq 0$$

$$\therefore \rho(A) = 2$$

Q Find Rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

Soln step 1. At least one minor of $3 \times 3 \neq 0$

$$|A|_{3 \times 3} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{vmatrix} = 2 \times (-3) \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$$\therefore \rho(A) < 3$$

step 2 At least one minor of $2 \times 2 \neq 0$

$$|A|_{2 \times 2} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$\therefore \rho(A) < 2$$

$$= \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$$

$$6 | 12 | 24$$

Step 3. At least one minor of $1 \times 1 \neq 0$

$$|A|_{1 \times 1} = 1 \neq 0 //$$

$\Rightarrow$ Hence, Rank of matrix A

$$\text{ie } \rho(A) = 1 //$$

Q Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$.

Soln Step 1 At least one minor of $3 \times 3 \neq 0$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -1 & 4 \\ 0 & 0 & 2 \end{vmatrix} \quad C_2 - C_3 = 2 \neq 0$$

$$\therefore \rho(A) = 3$$

Q Find the rank of matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Soln Step 1 At least one minor of $3 \times 3 \neq 0$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{vmatrix} \quad \begin{matrix} C_3 \rightarrow C_2 - C_3 \\ C_2 \rightarrow C_2 + 2C_3 \end{matrix} = \begin{vmatrix} 2 & 7 & 1 \\ 3 & 7 & 1 \\ -1 & 0 & 0 \end{vmatrix} = 0$$

$$\therefore \rho(A) < 3$$

Step 2

At least one minor of $2 \times 2 \neq 0$

$$|A|_{2 \times 2} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -3 \neq 0$$

$$\therefore \rho(A) = 2 //$$

II) Reducing in Normal Form :- (Applying Row and Column transformation)

• Normal form :- Matrix A reducing in Identity form using elementary row or column transformation.

Ex (i) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow I_4$ then $\boxed{\rho(A) = 4}$

In this case $\boxed{\rho(A) = 3}$

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \boxed{\rho(A) = 2}$

• If order is $m \times n$, so we change $(n-1)$ rows or column at one time.

Q Find the rank of matrix of A.

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Elementary transformation

Row	Column
• $R_i \leftrightarrow R_j$	• $C_i \leftrightarrow C_j$
• $R_i \rightarrow \frac{R_i}{k} \quad k \neq 0$ $\Rightarrow kR_i$	• $C_i \rightarrow \frac{C_i}{k}$ $\rightarrow kC_i$
• $R_1 \rightarrow R_1 \pm R_2 \pm R_3$	• $C_1 \rightarrow C_1 \pm C_2 \pm C_3$
• $R_1 \rightarrow R_1 \pm kR_2$	• $C_1 \rightarrow C_1 \pm kC_2$
• $R_1 \rightarrow k_1R_1 \pm k_2R_2$	• $C_1 \rightarrow k_1C_1 + k_2C_2$

$$\boxed{k_1, k_2, k \neq 0; \in \mathbb{R}}$$

Soln i) make a_{11} as unity, i.e. 1.

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(ii) with the help of R_1 make a_{21}, a_{31}, a_{41} are zero.

$$R_2 \rightarrow R_2 + 2R_1, R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(iii) with the help of C_1 make a_{12}, a_{13}, a_{14} are zero.

$$C_2 \rightarrow C_2 - 2C_1, C_3 \rightarrow C_3 - 3C_1, C_4 \rightarrow C_4 + C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{3}, R_3 \rightarrow \frac{R_3}{-2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(iv) without using R_1 and C_1 make other element zero if possible.

$$R_4 \rightarrow R_4 - R_3, R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(v)

$$C_4 \rightarrow C_4 + C_3, C_3 \rightarrow C_3 - C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$P(A) = 2$$

6/12/24

Q Find the rank of matrix A.

$$\Rightarrow A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 3 & 2 \\ -8 & -1 & -3 & -4 \end{bmatrix}$$

Soln. $A = \begin{bmatrix} 1 & 8 & 3 & 6 \\ 3 & 0 & 3 & 2 \\ -1 & -8 & -3 & -4 \end{bmatrix} \quad R_1 \leftrightarrow C_2$

$$= \begin{bmatrix} 1 & 8 & 3 & 6 \\ 0 & -24 & -6 & -16 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 - 3R_1 \mid R_3 \rightarrow R_3 + 4R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -24 & -6 & -16 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} C_2 \rightarrow C_2 - 8C_4 \mid C_3 \rightarrow C_3 + 3C_4 \\ C_4 \rightarrow C_4 - 6C_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{-8} \end{bmatrix} \quad \begin{array}{l} C_2 \rightarrow \frac{1}{-24}C_2 \mid C_3 \rightarrow \frac{1}{-6}C_3 \mid C_4 \rightarrow \frac{1}{-16}C_4 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \rightarrow -8R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} C_4 \rightarrow C_4 - C_2 \\ C_3 \rightarrow C_3 - C_2 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C_3 \leftrightarrow C_4$$

Hence

$$\rho(A) = 3$$

$$\text{11)} \quad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 1 & 5 \end{bmatrix}$$

$$\text{Soln: } A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 6R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \quad \begin{array}{l} C_2 \rightarrow C_2 - 2C_1 \\ C_3 \rightarrow C_3 - 3C_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad R_4 \rightarrow R_4 - R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \rightarrow R_4 - R_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \rightarrow \frac{1}{-4}C_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_3 \rightarrow C_3 + 8C_2 \\ C_4 \rightarrow C_4 + 3C_2 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \rightarrow \frac{C_2}{-3} \mid C_4 \rightarrow \frac{C_4}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \rightarrow C_4 - C_2$$

Hence $P(A) = 3$

$$\text{III} \rightarrow A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$\text{Sol}^n: A = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 7 \\ 1 & 1 & 1 & 12 \end{bmatrix} \quad \begin{array}{l} C_1 \rightarrow -C_1 + C_2 \\ C_2 \rightarrow -C_2 + C_3 \\ C_3 \rightarrow -C_3 + C_4 \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 12 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_2 \rightarrow R_2 - R_1 \\ R_1 \rightarrow R_1 - R_4 \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \\ C_4 \rightarrow C_4 - 12C_1 \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_1 \rightarrow R_1 + 7R_2 \\ R_2 \rightarrow R_2 + R_3 \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_4$$

Hence $\rho(A) = 2$

III Echelon Form (Row Echelon form)
i.e. Applying only Row transformation.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Working Rules:-

- ① Make a_{11} unity, i.e. 1
- ② With the help of R_1 make $a_{21}, a_{31}, a_{41}, \dots$ zero,

$$\rightarrow \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$$

- 1) All zeroes of non zero row before 1st element are in ascending order
- 2) Diagonal element should be unity i.e. 1
- 3) Now, No. of non zero rows are called RANK OF Matrix.

Q Find the rank of given matrix by Echelon form.

$$i) A = \begin{bmatrix} 0 & 2 & 3 & 1 & 4 \\ 1 & 1 & 1 & 4 & 3 \\ 0 & 3 & 8 & 6 & 1 \\ 3 & 4 & 4 & 2 & 1 \end{bmatrix}_{4 \times 5}$$

Soln

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 3 & 8 & 6 & 1 \\ 3 & 4 & 4 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

-with the help of R_1 make a_{21}, a_{31}, a_{41} zero.

$$R_3 \rightarrow R_3 - 2R_1 \quad | \quad R_4 \rightarrow R_4 - 3R_1$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 1 & 6 & -2 & -5 \\ 0 & 1 & 1 & 10 & -8 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 1 & 6 & -2 & -5 \\ 0 & 0 & -5 & -8 & -3 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 6 & -2 & -5 \\ 0 & 0 & -5 & -8 & -3 \end{bmatrix}$$

$$R_4 \rightarrow 2R_4 + R_3$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 6 & -2 & -5 \\ 0 & 0 & -1 & -21 & -20 \end{bmatrix}$$

$$R_4 \rightarrow 9R_4 + R_3$$

$$= \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 0 & 2 & 3 & 1 & 4 \\ 0 & 0 & 6 & -2 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \text{non. zero rows} = 4$$

$$\therefore \rho(A) = 4 //$$

$$\text{ii)} \quad A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Soln.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - 6R_1$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_4$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & +3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & +3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + R_3$$

$\therefore$ No. of non zero rows = 3

$$\therefore \text{rank}(A) = 3$$

$$\text{iii)} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix} = A$$

Soln

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_3$$

$$= \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_4 \rightarrow \frac{1}{5} R_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ no. of non-zero rows = 2

$$\therefore \text{rank}(A) = 2$$

Nullity of Matrix:-

If 'A' be a sq. matrix of order n (i.e. $n \times n$). Then nullity is defined as

$$N(A) = n - r \\ = n - \rho(A)$$

i) Example:-

$$[A]_{3 \times 3} ; \rho(A) = 2$$

$$\text{then, } N(A) = 3 - 2 = 1$$

ii) Example:-

$$[A]_{5 \times 5} ; \rho(A) = 3$$

$$\text{then, } N(A) = 5 - 3 = 2$$

To find solution of simultaneous linear equations:- (Applying only row transformation)

Let,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

no. of unknowns
(n) = (x, y, z) 3

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore AX = B$$

I. > To find Augmented matrix [ie $[A:B]$]

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$$

II. > To find Rank ie. $\{ \rho(A) = ? / \rho(A:B) = ? \}$

Some Cases Arise

i) $\rho(A) = \rho(A:B) = n$

→ The system is said to be consistent.

We get a unique solution (only one solⁿ)

As $AX = B$

ii) $\rho(A) = \rho(A:B) < n$

→ The system is said to be consistent

We get infinite solutions.

As $AX = B$

iii) $\rho(A) \neq \rho(A:B)$

→ The system is said to be inconsistent

We get no solution.

Non-Homogeneous eqⁿs.

If $b_1, b_2, b_3, \dots, b_n$ not all are zero simultaneously some may zero.

The system of equations is said to be

"Non-Homogeneous equations" system.

Homogeneous eqⁿ

If $b_1, b_2, b_3, \dots, b_n$ all are zero simultaneously.

Then system of equations is said to be "Homogeneous Equations" system.

Example: Test the consistency of equations: $5x+3y+4z=4$,
 $y+2z=1$, $x-y+2z=0$, $2x+y+6z=2$. It is consistent solve it.

Soln: let $A = \begin{bmatrix} 5 & 3 & 4 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

Now, Augmented matrix

$$[A:B] = \begin{bmatrix} 5 & 3 & 4 & : & 4 \\ 0 & 1 & 2 & : & 1 \\ 1 & -1 & 2 & : & 0 \\ 2 & 1 & 6 & : & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 5 & 3 & 4 & : & 4 \\ 2 & 1 & 6 & : & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 8 & 4 & : & 4 \\ 0 & 3 & 2 & : & 2 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 - 5R_2 \\ R_4 \rightarrow R_4 - 2R_2 \end{array}$$

$$= \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 2 & 1 & : & 1 \\ 0 & 3 & 2 & : & 2 \end{bmatrix} \quad R_3 \rightarrow \frac{1}{4}R_3$$

$$= \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -3 & : & -1 \\ 0 & 0 & -4 & : & 1 \end{bmatrix} \quad \begin{array}{l} R_3 \leftrightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 - 3R_2 \end{array}$$

$$= \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & -4 & : & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_4$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right]$$

$\underbrace{\hspace{10em}}_{A \quad B}$
 $\underbrace{\hspace{10em}}_{A:B}$

$$\rho(A) = 3 ; \rho(A:B) = 4$$

Hence, There is no solution exist.

Q Test the consistency of equations: $x+2y-z=3$; $3x-y+2z=1$,
 $2x-2y+3z=2$, $x-y+z=-1$

Soln let $[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array} \right] ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array} \right] \quad R_2 \rightarrow -R_2 + R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 + 6R_2 \\ R_4 \rightarrow R_4 + 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right] \quad R_3 \rightarrow \frac{R_3}{5}, R_4 \rightarrow \frac{R_4}{2}$$

$$\Rightarrow [A:B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & 1 & 0 & : & 4 \\ 0 & 0 & 1 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{A:B}$

Here, $\rho(A) = 3$; $\rho(A:B) = 3$; $n = 3$

$\therefore$ System is consistent

$AX = B$ has unique soln.

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

$$\rightarrow x + 2y - z = 3 \quad \text{--- (1)}$$

$\left. \begin{matrix} y = 4 \\ z = 4 \end{matrix} \right\}$ Put in (1)
 $x = -1$

Q Test the consistency of eqns. $5x + 3y + 7z = 4$
 $3x + 26y + 2z = 9$; $7x + 2y + 10z = 5$.

Soln

let

$$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \quad R_1 \rightarrow R_1 + R_2 - R_3$$

$$= \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & -55 & 5 & : & -15 \\ 0 & -187 & 17 & : & -51 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{matrix}$$

$$\Rightarrow [A:B] = \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & 11 & -1 & : & 3 \\ 0 & 11 & -1 & : & 3 \end{bmatrix} \quad R_2 \rightarrow \frac{R_2}{5} ; R_3 \rightarrow \frac{R_3}{17}$$

$$= \begin{bmatrix} 1 & 27 & -1 & : & 8 \\ 0 & 11 & -1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

Here, $\{ \rho(A) = 2 ; \rho(A:B) = 2 \} \wedge (n=3)$

$\therefore$ System is consistent,

We get infinite solⁿ as $AX=B$

$$\begin{bmatrix} 1 & 27 & -1 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 0 \end{bmatrix}$$

$$x + 27y - z = 8 \quad \text{--- (i)}$$

$$11y - z = 3 \quad \text{--- (ii)}$$

let $\begin{cases} z = k \\ \Rightarrow y = \frac{3+k}{11} \\ x = \frac{7-16k}{11} \end{cases}$

Put in (i)

Put $k=1, 2, 3, \dots$

We will get infinitely many solⁿs.

Q Test the consistency of given system of equations.

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

Solⁿ

let

$$[A:B] = \begin{bmatrix} 2 & 3 & 1 & : & 9 \\ 1 & 2 & 3 & : & 6 \\ 3 & 1 & 2 & : & 8 \end{bmatrix}$$

$$; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underline{n=3}$$

$$= \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 1 & : & 9 \\ 3 & 1 & 2 & : & 8 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1$$

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -5 & : & -3 \\ 0 & -5 & -7 & : & -10 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -5 & : & -3 \\ 0 & 0 & 18 & : & 5 \end{bmatrix} \quad R_3 \rightarrow R_3 - 5R_2$$

Here, $\rho(A) = 3$; $\rho(A:B) = 3$; $n = 3$

$\therefore AX = B$ has unique soln.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$$

$$x + 2y + 3z = 6 \quad \text{--- (i)}$$

$$-y - 5z = -3 \quad \text{--- (ii)}$$

$$18z = 5 \quad \text{--- (iii)}$$

$$\Rightarrow \boxed{z = \frac{5}{18}}$$

$$\Rightarrow y = -3 + \frac{5 \times 5}{18}$$

$$\Rightarrow \boxed{y = \frac{29}{18}}$$

$$\Rightarrow \text{from (i)} \quad x = 6 - \frac{2 \times 29}{18} - \frac{3 \times 5}{18}$$

$$\Rightarrow \boxed{x = \frac{35}{18}}$$

Q Test the consistency of equations: $x - y + z = 2$,
 $3x - y + 2z = -6$, $3x + y + z = -18$

Sol.

let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad B = \begin{bmatrix} 2 \\ -6 \\ -18 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 3 & -1 & 2 & : & -6 \\ 3 & 1 & 1 & : & -18 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 2 & -1 & : & -12 \\ 0 & 4 & -2 & : & -24 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 2 & 1 & : & -12 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_2$$

Here, $\rho(A) = 2$, $\rho(A:B) = 2$, $< n = 3$

$\therefore AX=B$ has unique solⁿ.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ 0 \end{bmatrix}$$

$$x - y + z = 2 \quad \text{--- (i)}$$

$$2y + z = -12 \quad \text{--- (ii)}$$

$$0 = 0 \quad \text{--- (iii)}$$

Put $\boxed{y = k}$
 $\boxed{z = -2(6+k)}$

from (i)

$$x + 2(6+k) + (-2(6+k)) = 2$$

$$\Rightarrow x = 2 + 12 + 2k + k$$

$$\boxed{x = 3k + 14} //$$

Test the consistency of eqns. $3x + 3y + 2z = 1$, $x + 2y = 4$,
 $10y + 3z = -2$, $2x - 3y - z = 5$.

Solⁿ

let

$$[A:B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 0 & 4 & : & -10 \\ 0 & 0 & 3 & : & -22 \\ 0 & 0 & -2 & : & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & 0 & 4 & : & -10 \\ 0 & 0 & 0 & : & -58 \\ 0 & 0 & 0 & : & 34 \end{bmatrix}$$

$$x+2y \dots \dots \dots$$

Pending

MMT

Q For what value of λ and μ the following
 $x+y+z=6, x+2y+3z=10, x+y+\lambda z=\mu$ have
 i) No solⁿ, ii) One solⁿ, iii) Infinite many solⁿ.

Solⁿ

$$\text{let } [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

i) For no solⁿ.

$$\rho(A) \neq \rho(A:B)$$

for which, take $\lambda-3=0 \mid \mu-10 \neq 0$

$$\text{ie } \lambda=3, \mu \neq 10$$

13/12/24

We have

$$\rho(A) = 2 \mid \rho(A:B) = 3 \quad \mid \rho(A) \neq \rho(A:B)$$

ii) For one soln.

$$\rho(A) = \rho(A:B) = (n=3)$$

For which take $\lambda - 3 \neq 0 \mid \mu - 10 = 0$

$$\text{i.e. } \boxed{\lambda \neq 3} \mid \boxed{\mu = 10}$$

We have

$$\boxed{\rho(A) = 3} \mid \rho(A:B) = 3$$

$$\rho(A) = \rho(A:B) = (n=3)$$

iii) For infinite soln.

$$\rho(A) = \rho(A:B) < (n=3)$$

For which take

$$\lambda - 3 = 0, \mu - 10 = 0$$

$$\boxed{\lambda = 3} \mid \boxed{\mu = 10}$$

We have

$$\{\rho(A) = 2 \neq \rho(A:B)\} \neq (n=3)$$

MMI

Q For what value of k , the following equations

$$x+y+z=1, \quad x+2y+4z=k, \quad x+4y+10z=k^2$$

have an soln, solve it in each case:

Soln

$$\text{Let } [A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & k \\ 1 & 4 & 10 & : & k^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 2 & 6 & : & k^2-k \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1; R_3 \rightarrow R_3 - R_2 \\ R_3 \rightarrow R_3 - 2R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 0 & 0 & : & k^2-3k+2 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_2$$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & k-1 \\ 0 & 0 & 0 & : & (k-1)(k-2) \end{bmatrix}$$

Here, $\rho(A) = 2$. It is clear that eqⁿ give infinite soln.

For infinite soln,

take, $(k-1)(k-2) = 0$

i.e., $k = 1, 2$.

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k-1 \\ (k-1)(k-2) \end{bmatrix}$$

For each case

i) $k = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x + y + z &= 1 & \text{--- (i)} \\ y + 3z &= 0 & \text{--- (ii)} \end{aligned}$$

let $\boxed{z = k_1}$
 $\boxed{y = -3k_1}$ Put in (i)
 $\boxed{x = 1 + 2k_1}$ $k_1 \in \mathbb{R}$

ii) $k = 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x + y + z &= 1 & \text{--- (i)} \\ y + 3z &= 1 & \text{--- (ii)} \end{aligned}$$

let $\boxed{z = k_2}$
 $\boxed{y = 1 - 3k_2}$ Put in (i)
 $\boxed{x = 2k_2}$ $k_2 \in \mathbb{R}$

Homogeneous Linear Equations :-

Let $a_{11}x + a_{12}y + a_{13}z = 0$ - (i)

$a_{21}x + a_{22}y + a_{23}z = 0$ - (ii)

$a_{31}x + a_{32}y + a_{33}z = 0$ - (iii)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $AX = B \Rightarrow AX = 0$

NOTE :- Homogeneous system is always consistent

i) If $\rho(A) = n \rightarrow$ no. of unknown (x, y, z)

System gives unique soln.

or trivial soln

or one soln.

ie $x=y=z=0$

here $|A| \neq 0$

ii) If $\rho(A) < n$, system gives non trivial soln.

or non-zero soln

or infinite soln

here $|A| = 0$ always

Q Find the value of λ for which the system of equations have a non-trivial solution.

P) $(3\lambda - 8)x + 3y + 3z = 0$

$3x + (3\lambda - 8)y + 3z = 0$

$3x + 3y + (3\lambda - 8)z = 0$

Soln 1.

$$\text{let } A = \begin{bmatrix} 3\lambda-8 & 3 & 3 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

System give non-trivial soln.
when $|A| = 0$

$$\text{i.e.; } \begin{vmatrix} 3\lambda-8 & 3 & 3 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0 \quad R_1 \rightarrow R_1 + R_2 + R_3$$

$$\Rightarrow (3\lambda-2) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0$$

$$\Rightarrow (3\lambda-2) \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3\lambda-11 & 0 \\ 0 & 0 & 3\lambda-11 \end{vmatrix} = 0 \quad \begin{array}{l} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \end{array}$$

$$\Rightarrow (3\lambda-2) \{ (3\lambda-11)^2 - 0 \} = 0$$

$$\Rightarrow \lambda = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$$

Q.11

$$\begin{array}{l} x + \lambda y + 3z = 0 \\ 4x + 3y + \lambda z = 0 \\ 2x + y + 2z = 0 \end{array}$$

Soln

$$\text{let } A = \begin{bmatrix} 1 & \lambda & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AB \quad AX = 0$$

If system of eqn. has trivial soln.

$$\Rightarrow |A| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & \lambda & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & \lambda & 3 \\ 0 & 4\lambda-3 & 12-\lambda \\ 0 & 2\lambda-1 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 4(4\lambda-3) - (12-\lambda)(2\lambda-1) = 0$$

$$\Rightarrow 16\lambda - 12 - 24\lambda + 12 + 2\lambda^2 - \lambda = 0$$

$$\Rightarrow 2\lambda^2 + (-9)\lambda = 0$$

$$\Rightarrow \lambda(2\lambda-9) = 0$$

$$\Rightarrow \boxed{\lambda=0}, \boxed{\lambda=9/2}$$

Q Find value of k for given system of equations has non zero soln. $2x+y+2z=0$, $x+y+3z=0$, $4x+3y+kz=0$

Soln. $\because$ given system of equations has non zero soln

$$\therefore |A| = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & -1 & -4 \\ 1 & 1 & 3 \\ 0 & -1 & k-12 \end{vmatrix} = 0$$

$$\Rightarrow -1[-(k-12) - (-4)] = 0$$

$$\Rightarrow \boxed{k=8}$$

$$\text{let } A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot X = O$$

Q Solve. $3x - y + z = 0$, $-15x + 6y - 5z = 0$, $5x - 2y + 2z = 0$.

Soln

Here,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To find rank

$$R_1 \rightarrow 2R_1 - R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -5 \\ 0 & -2 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 + 15R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} R_3 \rightarrow 3R_3 + R_2$$

$$\therefore \rho(A) = 3 = (n=3)$$

$\therefore$ System gives only one soln:

$$\text{i.e. } \boxed{x=0}, \boxed{y=0}, \boxed{z=0}$$

Q Solve. $2w + 3x - y - z = 0$

$$4w - 6x - 2y + 2z = 0$$

$$-6w + 12x + 3y - 4z = 0$$

Soln

Let.

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix}; X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}; O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 21 & 0 & -7 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{array}$$

$$= \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow \frac{R_2}{-4} \\ R_3 \rightarrow \frac{R_3}{-7} \end{array}$$

$$= \begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

$$\rho(A) = 2 < (n=4)$$

System gives non-zero infinite soln.

$$A\vec{x}, AX=0$$

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2w + 3x - y + z = 0 \quad \text{--- (i)}$$

$$3x - z = 0 \quad \text{--- (ii)}$$

$$\text{let } \boxed{x = k_1}$$

$$3k_1 - z = 0 \Rightarrow \boxed{z = 3k_1}$$

$$\text{let } \boxed{w = k_2}$$

$$2k_2 + 3k_1 - y - 3k_1 = 0$$

$$\boxed{y = +2k_2}$$

Thus

$$w = k_2, x = k_1, y = 2k_2, z = 3k_1$$

$$\text{Put } k_1, k_2 = 0, 1, 2, 3 \dots$$

$$\{k_1, k_2 \in \mathbb{R}\}$$