

UNIT - I

Differential Calculus : $\rightarrow$

Standard Results : $\rightarrow$

$$1) \frac{d}{dx} (x^n) = nx^{n-1}$$

$$2) \frac{d}{dx} (e^x) = e^x$$

$$3) \frac{d}{dx} (a^x) = a^x \log_e a$$

$$4) \frac{d}{dx} (\log_e x) = \frac{1}{x}$$

$$5) \frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e$$

$$6) \frac{d}{dx} (\sin x) = \cos x$$

$$7) \frac{d}{dx} (\cos x) = -\sin x$$

$$8) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$9) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$10) \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$11) \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$$

$$12) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$13) \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$14) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$15) \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$16) \frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$17) \frac{d}{dx} (\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$18) \frac{d}{dx} (u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$19) \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Leibnitz's theorem: $\rightarrow$ If u and v are two functions of x , then

$$D^n(uv) = D^n u \cdot v + nC_1 D^{n-1} u \cdot Dv + nC_2 D^{n-2} u \cdot D^2 v + \dots + nC_r D^{n-r} u \cdot D^r v + \dots + u D^n v$$

Expansion of functions by Maclaurin's & Taylor Theorem

Maclaurin's Theorem: $\rightarrow$ If $f(x)$ be a function of the variable x such that it can be expanded in ascending powers of x and the expansion be differentiable any number of times, then the theorem states that

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

Proof: $\rightarrow$ Let $f(x) = A_0 + A_1 x + A_2 x^2 + \dots \rightarrow (1)$

Where $A_0, A_1, A_2, \dots$ are constants

Now by successive differentiation of eq. (1)

w.r.t. x we have

$$f'(x) = A_1 \cdot 1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \dots$$

$$f''(x) = 2A_2 + 3 \cdot 2 A_3 x + 4 \cdot 3 A_4 x^2 + \dots$$

$$f'''(x) = 3 \cdot 2 \cdot A_3 + 4 \cdot 3 \cdot 2 A_4 x + \dots$$

Substituting $x=0$ in each of above relations we get

$$f(0) = A_0, \quad f'(0) = A_1,$$

$$f''(0) = 2! A_2, \quad f'''(0) = 3! A_3$$

$$+ \dots + \dots + \dots$$

$$A_0 = f(0)$$

$$A_1 = f'(0)$$

$$A_2 = \frac{f''(0)}{2!}$$

$$A_3 = \frac{f'''(0)}{3!}$$

+ - - - -

Substituting the values of A_0, A_1, A_2 in eq (1), we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

Q4. Expand by following by Maclaurin's theorem

1) e^x

2) $(a+x)^m$

3) $\log(1+x)$

4) $\sin x$

5) $\tan x$

6) $\sec x$

Sol: $\rightarrow$ 1) Let $f(x) = e^x$

$$f'(x) = e^x$$

$$f(0) = 1$$

$$f''(x) = e^x$$

$$f'(0) = 1$$

$$f^n(x) = e^x$$

$$f''(0) = 1$$

By Maclaurin's theorem we get, $f^n(0) = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + x^n + \dots$$

$$\text{ii) } f(x) = (a+x)^m, \quad f(0) = a^m$$

$$f'(x) = m(a+x)^{m-1}, \quad f'(0) = m a^{m-1}$$

$$f''(x) = m(m-1)(a+x)^{m-2}, \quad f''(0) = m(m-1)a^{m-2}$$

$$\vdots$$

$$f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)(a+x)^{m-n}$$

$$f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)a^{m-n}$$

Now by Maclaurin's theorem, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$(a+x)^m = a^m + x \cdot m a^{m-1} + \frac{x^2}{2!} m(m-1)a^{m-2} + \cdots + \frac{x^n}{n!} m(m-1)(m-2)\cdots(m-n+1)a^{m-n} + \cdots$$

$$\text{3) } f(x) = \log(1+x), \quad f(0) = 0$$

$$f'(x) = (1+x)^{-1}, \quad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2}, \quad f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3}, \quad f'''(0) = 2 = 2!$$

$$f^{(4)}(x) = -2 \times 3 (1+x)^{-4}, \quad f^{(4)}(0) = -3!$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n-1} (n-1)! (1+x)^{-n}$$

$$f^{(n)}(0) = (-1)^{n-1} (n-1)!$$

By Maclaurin's theorem, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$

Ans.

$$\begin{aligned}
 4) \quad f(x) &= \sin x & f(0) &= 0 \\
 f'(x) &= \cos x & f'(0) &= 1 \\
 f''(x) &= -\sin x & f''(0) &= 0 \\
 f'''(x) &= -\cos x & f'''(0) &= -1 \\
 f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \dots \\
 \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 5) \quad f(x) &= \tan x = y & f(0) &= 0 \\
 y_1 = f'(x) &= \sec^2 x \\
 &= 1 + \tan^2 x = 1 + y^2
 \end{aligned}$$

$$y_2 = 2y y_1$$

$$y_3 = 2(y y_2 + y_1^2)$$

$$y_4 = 2(y y_3 + y_1 y_2 + 2y_1 y_2)$$

$$y_5 = 2(y y_4 + y_1 y_3 + 3y_1 y_2 + 3y_1^2)$$

$$(y_1)_0 = 1 + 0 = 1$$

$$(y_2)_0 = 2y_0 (y_1)_0 = 0$$

$$(y_3)_0 = 2, \quad y_4(0) = 0, \quad y_5(0) = 16$$

$$y = \tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \dots$$

Ans

$$6) y = \sec x$$

$$y_1 = \sec x \tan x = y \tan x$$

$$y_2 = y \sec^2 x + y_1 \tan x = y^3 + y_1 \tan x$$

$$y_3 = 3y^2 y_1 + y_1 \sec^2 x + y_2 \tan x$$

$$= 4y^2 y_1 + y_2 \tan x$$

$$y_4 = 8y y_1^2 + 4y^2 y_2 + y_3 \tan x + y_2 \sec^2 x$$

$$= 8y y_1^2 + 5y^2 y_2 + y_3 \tan x$$

$$(y)_0 = 1, (y_1)_0 = 0, (y_2)_0 = 1, (y_3)_0 = 0$$

$$(y_4)_0 = 5$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \dots$$

Q4

obtain by Maclaurin's theorem the first five terms in the expansion of $e^{\sin x}$.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

$$\text{Sol} \rightarrow y = e^{\sin x}, (y)_0 = e = 1$$

$$y_1 = \cos x \cdot e^{\sin x} = y \cos x, (y_1)_0 = (y_0)' = 0$$

$$y_2 = y_1 \cos x - y \sin x, (y_2)_0 = (y_1)'_0 = 0$$

$$y_3 = y_2 \cos x - y_1 \sin x - y \cos x$$

$$= (y_2 - y) \cos x - 2y_1 \sin x$$

$$\therefore (y_3)_0 = [(y_2 - y)_0] - 0 = (1 - 1) = 0$$

$$y_4 = (y_3 - y_1) \cos x - (y_2 - y) \sin x - 2y_2 \sin x - 2y_1 \cos x$$

$$= (y_2 - 3y_1) \cos x - (3y_2 - y_1) \sin x$$

$$\therefore (y_4)_0 = (0 - 3 \cdot 1) \cdot 1 - 0 = -3$$

$$y_5 = (y_4 - 3y_2) \cos x - (y_3 - 3y_1) \sin x$$

$$- (3y_3 - y_1) \sin x - (3y_2 - y_1) \cos x$$

$$= (y_4 - 6y_2 + y_1) \cos x - (4y_3 - 4y_1) \sin x$$

$$(y_5)_0 = (-3 - 6 \cdot 1) \cdot 1 - 0 = -9$$

$$y = (y)_0 + x (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \dots$$

$$y = 1 + x + \frac{x^2}{2!} - 3 \cdot \frac{x^4}{4 \cdot 3 \cdot 2} - 9 \cdot \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2} + \dots$$

$$y = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{18} + \dots$$

Ans.

Ex(68)

Taylor's Theorem $\rightarrow$ If $f(a+h)$ be a function of the variable h such that it can be expanded in ascending powers of h and this expansion be differentiable any number of times then the theorem states that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

proof $\rightarrow f(a+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \dots + A_n h^n + \dots \rightarrow (1)$

Where $A_0, A_1, A_2, \dots, A_n$ are constant

Now by successive diff. of (1) w.r. to h

$$f'(a+h) = A_1 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3 + \dots$$

$$f''(a+h) = 2A_2 + 3 \cdot 2A_3 h + 4 \cdot 3A_4 h^2 + \dots$$

$$f'''(a+h) = 3 \cdot 2A_3 + 4 \cdot 3 \cdot 2A_4 h + \dots$$

Put $h=0$ in each of above relations

$$f(a) = A_0, \quad f'(a) = A_1, \quad f''(a) = 2! A_2$$

$$f'''(a) = 3! A_3 \quad \dots \quad f^n(a) = n! A_n$$

$$A_0 = f(a), \quad A_1 = f'(a), \quad A_2 = \frac{f''(a)}{2!}$$

$$A_3 = \frac{f'''(a)}{3!} \quad \dots \quad A_n = \frac{f^n(a)}{n!}$$

Substituting these values of $A_0, A_1, A_2, \dots$ in (1)

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

Ques Expand $\sin x$ in powers of $(x - \frac{\pi}{2})$

Sol $\rightarrow$ Let $f(x) = \sin x$ we can write

$$f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = f(a+h)$$

here $a = \frac{\pi}{2}$, $h = x - \frac{\pi}{2}$

By Taylor's theorem, we have

$$f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = f\left(\frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \dots \rightarrow (1)$$

Now $f(x) = \sin x$	$f(\pi/2) = \sin \pi/2 = +1$
$f'(x) = \cos x$	$f'(\pi/2) = \cos \pi/2 = 0$
$f''(x) = -\sin x$	$f''(\pi/2) = -1$
$f'''(x) = -\cos x$	$f'''(\pi/2) = 0$
$f^{iv}(x) = \sin x$	$f^{iv}(\pi/2) = 1$

Putting these values in (i), we get

$$\sin x = 1 + \left(x - \frac{\pi}{2}\right) \cdot 0 + \left(x - \frac{\pi}{2}\right)^2 \frac{(-1)}{2!} +$$

$$\frac{\left(x - \frac{\pi}{2}\right)^3}{3!} \cdot 0 + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} (1) + \dots$$

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \dots$$

Ans.

Q4 Expand $2x^3 + 7x^2 + x - 1$ in powers of $(x-2)$ by Taylor's theorem.

Sol $\rightarrow$ Let $f(x) = 2x^3 + 7x^2 + x - 1 \rightarrow \textcircled{1}$

$$f(x) = f[2 + (x-2)] = f(a+h)$$

here $a=2$, $h=x-2$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \dots \rightarrow \textcircled{2}$$

$$f'(x) = 6x^2 + 14x + 1$$

$$f''(x) = 12x + 14$$

$$f'''(x) = 12$$

$$f^{IV}(x) = 0$$

put $x=2$ in above, we get

$$f(2) = 2 \cdot 2^3 + 7 \cdot 2^2 + 2 - 1 = 45$$

$$f'(2) = 53$$

$$f''(2) = 38, \quad f'''(2) = 12, \quad f^{iv}(2) = 0$$

$$f(x) = 45 + (x-2) \cdot 53 + \frac{(x-2)^2}{2!} \cdot 38 + \frac{(x-2)^3}{3!} \cdot 12 + 0$$

$$f(x) = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Partial Differentiation $\therefore \rightarrow$

Qy. if $u = \log \left(\frac{x^2+y^2}{xy} \right)$ verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Sol $\rightarrow u = \log(x^2+y^2) - \log(xy)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x - \frac{1}{xy} \cdot y$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2x}{(x^2+y^2)^2} \cdot 2y = \frac{-4xy}{(x^2+y^2)^2} \quad (1)$$

Again $\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{xy} \cdot x$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-2y}{(x^2+y^2)^2} \cdot 2x = \frac{-4xy}{(x^2+y^2)^2} \quad (2)$$

Hence from (1) & (2)

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Q4. If $u = (x^2 + y^2 + z^2)^{-1/2}$ prove that
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -u$$

Sol $\rightarrow$ $u = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-\frac{1}{2} - 1} \cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} =$$

$$\frac{-x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(-y^2)}{(x^2 + y^2 + z^2)^{3/2}} + \frac{(-z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$
$$= -\frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= - (x^2 + y^2 + z^2)^{1 - \frac{3}{2}}$$

$$= - (x^2 + y^2 + z^2)^{-1/2} = -u$$

Homogeneous function:

A function in which every term is of the same degree is known as a homogeneous function of that degree. Thus every homogeneous function of x and y of degree n can be expressed in the form

$$x^n F\left(\frac{y}{x}\right)$$

Euler's theorem on homogeneous functions

Theorem: $\rightarrow$ If u is a homogeneous function of x and y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Proof: $\rightarrow$ Since u is a homogeneous function of x and y of degree n .

Since $u = x^n F\left(\frac{y}{x}\right) \rightarrow (1)$

diff. w.r. to x we get

$$\frac{\partial u}{\partial x} = nx^{n-1} F\left(\frac{y}{x}\right) + x^n F'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right)$$

$$x \frac{\partial u}{\partial x} = nx^n F\left(\frac{y}{x}\right) - x^{n-1} y F'\left(\frac{y}{x}\right)$$

$$x \frac{\partial u}{\partial x} = nu - x^{n-1} y F'\left(\frac{y}{x}\right) \rightarrow (2)$$

again diff (1), we get

$$\frac{\partial y}{\partial y} = x^n F' \left(\frac{y}{x} \right) \left(\frac{1}{x} \right)$$

$$y \frac{\partial y}{\partial y} = x^{n-1} y F' \left(\frac{y}{x} \right) \rightarrow (3)$$

adding eq (2) and (3), we get

$x \frac{\partial y}{\partial x} + y \frac{\partial y}{\partial y} = nu$
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Q4. If $u = \tan^{-1} \left(\frac{x^3 + y^3}{x+y} \right)$, then prove

$$\text{that } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Sol $\rightarrow$ Let $f = \tan u = \frac{x^3 + y^3}{x+y}$

It is a homogeneous function of degree 2
hence By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

On putting the value of f

$$x \frac{\partial}{\partial x} \tan u + y \frac{\partial}{\partial y} \tan u = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \frac{2 \sin u \times \cos^2 u}{\cos u}$$

$$= 2 \sin u \cos u$$

$$= \sin 2u$$

Maxima and minima of function of two variables

Working Rule: $\rightarrow$ If function $f(x)$ be given then find $f'(x)$ and equate it to zero. Solve this equation for x .

Let its roots be $a_1, a_2, a_3, \dots$. find $f''(x)$ and hence find $f''(a_1), f''(a_2), f''(a_3), \dots$

If $f''(a_1)$ is negative, then we have a maximum at $x = a_1$.

If $f''(a_1)$ is positive, then we have a minimum at $x = a_1$.

If $f''(a_1) = 0$ find $f'''(x)$ and then $f'''(a_1)$.

If $f'''(a_1) \neq 0$ then there is neither maxima nor minima at $x = a_1$.

If $f'''(a_1) = 0$ find $f^{(4)}(x)$ and then $f^{(4)}(a_1)$.

If $f^{(4)}(a_1)$ is negative then $f(x)$ is maximum at $x = a_1$ and if $f^{(4)}(a_1)$ is positive then $f(x)$ is minimum at $x = a_1$.

If $f^{(4)}(a_1) = 0$ then find $f^{(5)}(a_1)$ and so on.

Maxima and minima in a closed interval $[a, b]$

Let $f(x)$ be a given function defined in $[a, b]$, let the minimum value of $f(x)$ be m and let maximum value of $f(x)$ be M then.

- ① Minimum value of $f(x)$ in $[a, b]$ is the smallest of m , $f(a)$ and $f(b)$.
- ② Maximum value of $f(x)$ in $[a, b]$ is the greatest of M , $f(a)$ and $f(b)$.

Q4. Find the largest and smallest values of $x^3 - 18x^2 + 96x$ in the interval $[0, 9]$

Sol $\rightarrow$ Given $f(x) = x^3 - 18x^2 + 96x$

$$f'(x) = 3x^2 - 36x + 96$$

$$f''(x) = 6x - 36$$

for maxima and minima, we put $f'(x) = 0$

$$\Rightarrow 3x^2 - 36x + 96 = 0$$

$$\Rightarrow x - 12x + 32 = 0$$

$$\Rightarrow (x - 8)(x - 4) = 0$$

$$\Rightarrow x = 4 \text{ and } 8 \text{ (critical points)}$$

At point $x = 4$

$$\Rightarrow f(x) \text{ has max. value at } x = 4$$

$$\therefore [f(x)]_{x=4} = 4^3 - 18(4^2) + 96(4) \\ = 64 - 288 + 384 \\ = 160$$

at $x=8$

$$f''(x=8) = 6(8) - 36 = +ve > 0 \\ \Rightarrow f(x) \text{ has min value at } x=8$$

$$[f(x)]_{x=8} = 8^3 - 18(8^2) + 96(8) \\ = 512 - 1152 + 768 = 128$$

also given $0 \leq x \leq 9$, i.e.

At point $x=0$ then $[f(x)]_{x=0} = 96$

At point $x=9$ then $[f(x)]_{x=9} \\ = 729 - 1458 + 864 = 135$

Hence min. value is 96 at $x=0$ and
max. value is 160 at $x=4$

Qy Find the maximum and minimum values of the function $f(x) = x^2 - 4x + 5$ in interval $1 \leq x \leq 4$.

Solu. $f(x) = x^2 - 4x + 5 \rightarrow \textcircled{1}$

$f'(x) = 2x - 4 \rightarrow \textcircled{2}$

for maxima and minima $f'(x) = 0$

$2x - 4 = 0 \Rightarrow x = 2$ (critical point)

Now $f''(x) = 2 \Rightarrow f''(2) = 2 > 0$

Hence function $f(x)$ has minimum value at $x=2$

$[f(x)]_{x=2} = 4 - 8 + 5 = 1$

also given in interval $1 \leq x \leq 4$ then

$$f(1) = 1 - 4 + 5 = 2 \text{ and}$$

$$f(4) = 16 - 16 + 5 = 5$$

thus maximum value of $f(x)$ at $x=4$ is 5.

Maxima and Minima of A Function of Two Variables

Working Rule $\rightarrow$

- 1) find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ or (f_x and f_y)
- 2) A necessary condition for maximum or minimum value is
$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$
- 3) Solve the above simultaneous equations
Let $(a_1, b_1), (a_2, b_2)$ -- be the solution of these equations
- 4) find $\frac{\partial^2 f}{\partial x^2} = r, \quad \frac{\partial^2 f}{\partial x \partial y} = s, \quad \frac{\partial^2 f}{\partial y^2} = t$
at these points,
- 5) A sufficient condition for maximum or minimum value is $rt - s^2 > 0$
- 6) (a) If $r > 0$ at one or more points then function f is minimum.

- ⑥ If $\gamma < 0$ at one or more points the function f is maximum.
- 7) If $\gamma - S^2 = 0$. Such case is undecided and further investigation is needed.
- 8) If $\gamma = 0$ nothing can be said about the maxima or minima. It requires further investigation.

~~***~~ ① A maximum or minimum value of a given function is called extreme value.

② $f(a, b)$ is to be a stationary value of $f(x, y)$, If $f_x(a, b) = 0$ and $f_y(a, b) = 0$ thus every extreme value is stationary value but converse may not be true.

Q4 find the radius of curvature at the point t of the following curve
 $x = a \cos^3 t$, $y = a \sin^3 t$ or $x^{2/3} + y^{2/3} = a^{2/3}$
 OR

prove that the radius of curvature at any point on the hypocycloid
 $x^{2/3} + y^{2/3} = a^{2/3}$ is three times the

perpendicular from the origin to the tangent at that point.

Sol: → The given equation of curve is

$$x^{2/3} + y^{2/3} = a^{2/3} \rightarrow (1)$$

the parametric eq of curve (1) are

$$x = a \cos^3 t, \quad y = a \sin^3 t \rightarrow (2)$$

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt} \right) / \left(\frac{dx}{dt} \right) = -\tan t$$

$$\text{again differentiate } \frac{d^2 y}{dx^2} = -\sec^2 t \left(\frac{dt}{dx} \right)$$

$$= -\sec^2 t \cdot \frac{1}{-3a \cos^2 t \sin t} = \frac{\sec^4 t \operatorname{cosec} t}{3a}$$

We know that

$$\text{radius of curvature } \rho = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}$$

$$\frac{d^2 y}{dx^2}$$

$$\rho = \frac{(1 + \tan^2 t)^{3/2}}{\left(\frac{1}{3a}\right) \sec^4 t \operatorname{cosec} t}$$

$$\rho = 3a \sin t \cos t \rightarrow (3)$$

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\frac{\sin t}{\cos t}$$

Equation of tangent at point t is

$$\Rightarrow y - a \sin^3 t = \left(\frac{dy}{dx}\right) (x - a \cos^3 t)$$

$$\Rightarrow y - a \sin^3 t = -\frac{\sin t}{\cos t} (x - a \cos^3 t)$$

$$\Rightarrow y \cos t - a \sin^3 t \cos t = -x \sin t + a \sin t \cos^3 t$$

$$\Rightarrow x \sin t + y \cos t = a \sin t \cos t (\sin^2 t + \cos^2 t)$$

$$\Rightarrow x \sin t + y \cos t = a \sin t \cos t \rightarrow (4)$$

which is equation of ~~tag~~ tangent line.
Now perpendicular length P from the origin $(0,0)$ to the tangent line (4) is

$$P = \frac{|0 + 0 - a \sin t \cos t|}{\sqrt{\sin^2 t + \cos^2 t}} = a \sin t \cos t$$

Hence (3) becomes

$$P = a \sin t \cos t$$

$$P = 3p$$

ROLLE'S THEOREM : $\rightarrow$

Statement : If $f(x)$ is a function of the variable x such that

- i) $f(x)$ is continuous in the closed interval $[a, b]$.
- ii) $f(x)$ is differentiable for every point in the open interval (a, b) i.e. $f'(x)$ exists for every point in the open interval (a, b) and
- iii) $f(a) = f(b)$ [or as a special case $f(a) = f(b) = 0$] then there is at least one point $x = c$ in the open interval (a, b) such that $f'(c) = 0$.

proof : condition (i) shows that the function $f(x)$ does not break at any point in the closed interval $[a, b]$.
condition (iii) i.e. $f(a) = f(b)$ shows that either (a) $f(x)$ is constant in the interval $[a, b]$ or (b) $f(x)$ should either increase or decrease as x takes values from a to b .

Q1 Verify Rolle's theorem where
 $f(x) = 2x^3 + x^2 - 4x - 2$

Sol: Here $f(x)$ is a rational integral function of x , therefore $f(x)$ is continuous and differentiable for all real values of x . Hence condition (i) and (ii) of Rolle's theorem are satisfied in any interval.

$$\text{Now } f(x) = 2x^3 + x^2 - 4x - 2$$

$$f(x) = x^2(2x+1) - 2(2x+1)$$

$$(2x+1)(x^2-2)$$

$$f(x) = 0 \Rightarrow (x^2-2)(2x+1) = 0$$

$$x = -\frac{1}{2}, \sqrt{2}, -\sqrt{2}$$

$$\therefore f\left(-\frac{1}{2}\right) = f(\sqrt{2}) = f(-\sqrt{2}) = 0$$

$f'(x)$ exists for all above values of x .

Now we take the interval $[-\sqrt{2}, \sqrt{2}]$

So that all conditions of Rolle's theorem are satisfied in this interval. We have to prove that there exists at least one

point $x=c$ in the open interval $(-\sqrt{2}, \sqrt{2})$ at which $f'(x) = 0$

$$\text{Now } f'(x) = 6x^2 + 2x - 4 = 0$$

$$\Rightarrow 3x^2 + x - 2 = 0$$

$$\Rightarrow (3x-2)(x+1) = 0$$

$$\Rightarrow x = -1, \frac{2}{3}$$

$$-\sqrt{2} < -1 < \sqrt{2} \quad \text{and} \quad -\sqrt{2} < \frac{2}{3} < \sqrt{2}$$

$$c = -1, \frac{2}{3}$$

Hence Rolle's theorem is verified.

Q4: Verify Rolle's theorem for the function $f(x) = x^3 - 12x$ in the interval $0 \leq x \leq 2\sqrt{3}$

Sol $\rightarrow$ Here $f(x) = x^3 - 12x$

$$f(x) = 0 \Rightarrow x^3 - 12x = 0$$

$$\Rightarrow x(x^2 - 12) = 0$$

$$\Rightarrow x = 0, \pm 2\sqrt{3}$$

Now $f(x)$ is a rational integral function of x , therefore $f(x)$ is continuous and differentiable for all real values of x . So $f(x)$ is

- (i) Continuous in the closed interval $[0, 2\sqrt{3}]$
- (ii) Differentiable in the open interval $]0, 2\sqrt{3}[$

$$\text{also } f(0) = f(2\sqrt{3}) = 0$$

Thus all condition of Rolle's theorem are satisfied. Consequently there exists at least one point ($x = c$) in the open interval $]0, 2\sqrt{3}[$ at which $f'(x) = 0$

$$f'(x) = 3x^2 - 12 = 0$$

$$x = \pm 2$$

$$\because 0 < 2 < 2\sqrt{3} \therefore c = 2$$

UNIT - 5

Matrices

Some elementary Concepts: $\rightarrow$

A Set of m & n numbers (real or complex) arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or $m \times n$ matrix. Numbers constituting the matrix are called elements of matrix.

Example: $\rightarrow \begin{bmatrix} 2 & 1 & 7 \\ 3 & 6 & 4 \end{bmatrix}$ is a 2×3 matrix and it has 6 elements.

Similarly

a_{11}	a_{12}	a_{13}	a_{1n}
a_{21}	a_{22}	a_{23}	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{m1}	a_{m2}	a_{m3}	a_{mn}

 is $m \times n$ matrix

any element a_{ij} of the matrix belongs to the i th row & j th column. A matrix is not a number. It has got no numerical value. A matrix is represented as $A = [a_{ij}]_{m \times n}$

Types of Matrices : $\rightarrow$ Let $[a_{ij}]_{m \times n}$ be a matrix of order $m \times n$, then for different values of m and n , we get different kinds of matrices.

1) Square Matrix : $\rightarrow$ When the no. of rows and columns are equal of a matrix. i.e. $m = n$ then the matrix is called a square matrix.
For ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 0 & 9 \end{bmatrix} \text{ is a square matrix of order } 3 \times 3.$$

2) Rectangular matrix : $\rightarrow$ When the no. of rows are not equal to no. of columns i.e. $m \neq n$ then the matrix is called a rectangular matrix.
For ex.

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 9 & 4 & 8 \end{bmatrix} \text{ is a rectangular matrix of order } 2 \times 3.$$

3) Row matrix : $\rightarrow$ If a matrix contains only one row then it is called a row matrix.
 $A = [1 \ 9 \ 5]$ is a row matrix of order 1×3 .

4) Column matrix : $\rightarrow$ If in a matrix, there is only one column, then it is called as Column matrix.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ is a column matrix of order } 4 \times 1.$$

5) Null matrix : $\rightarrow$ If all the elements of a $m \times n$ matrix are zero, then it is called a null or zero matrix. Usually it is denoted by $O_{m \times n}$.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a } 2 \times 3 \text{ null matrix.}$$

6) Horizontal Matrix : $\rightarrow$ If $(m \times n)$ $m < n$ i.e. when no. of rows is less than the no. of columns then the matrix is called a horizontal matrix.

$$A = \begin{bmatrix} 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix} 2 \times 4$$

7) Vertical Matrix : $\rightarrow$ If $m > n$ then the matrix is called as vertical matrix.

$$A = \begin{bmatrix} 2 & 4 \\ 9 & 6 \\ 1 & 3 \end{bmatrix}$$

Scalar Matrix : $\rightarrow$ If all elements of leading diagonal of diagonal matrix are equal then it is called Scalar matrix or a square matrix $A = [a_{ij}]$ is called a Scalar matrix if $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = k$ (say) when $i = j$

ex. : $\rightarrow \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a Scalar matrix of order 3×3 .

Unit Matrix : $\rightarrow$ If all elements of leading diagonal of diagonal matrix are equal to 1, then it is called a unit matrix.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3×3

Diagonal matrix : $\rightarrow$ If all elements excluding the elements $a_{11}, a_{22}, a_{33} \dots a_{nn}$ of leading diagonal of square matrix are zero, then the matrix is called diagonal matrix. Hence a square matrix $A = [a_{ij}]_{m \times n}$ is a diagonal matrix if $a_{ij} = 0$ when $i \neq j$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ 3×3

Unit Matrix : $\rightarrow$ If all the elements of leading diagonal of a diagonal matrix are equal to 1, then it is called a unit matrix. Unit matrices of order $1 \times 1, 2 \times 2, \dots, n \times n$ are denoted respectively by $I_1, I_2, \dots, I_n$. A unit matrix I_n of order $n \times n$ is also called a unit matrix of order n .

$$I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

Triangular matrix : $\rightarrow$

a) **Upper triangular matrix** : $\rightarrow$ A square matrix in which all elements below the diagonal are zero is called upper triangular matrix.

b) **Lower triangular matrix** : $\rightarrow$ A square matrix in which all elements above the leading diagonal are zero, is called lower triangular matrix.

Singular matrix: $\rightarrow$ A matrix whose det. is zero.

i.e. $|A| = 0$

Non-Singular matrix: $\rightarrow$ A matrix whose det. is not equal to zero, i.e. $|A| \neq 0$.

Symmetric matrix: $\rightarrow$ Whose transposition of matrix is equal to it.

i.e. $A^T = A$ or $a_{ij} = a_{ji}$

The value of determinant of A is same as the value of determinant of its transpose.

Ex. Transpose of $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 9 & 7 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 9 & 7 \\ 5 & 7 & 11 \end{bmatrix}$

and

transpose of $A = \begin{bmatrix} 4 & 3 & 7 \\ 2 & 1 & 5 \\ 6 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 1 & 0 \\ 7 & 5 & 9 \end{bmatrix}$

Skew Symmetric $\rightarrow$ Whose transpose is equal to its -ve matrix.

i.e $A^T = -A$, it means $a_{ij} = -a_{ji}$

Comparable Matrix $\rightarrow$ example

if $A=B$ then

$$A = \begin{bmatrix} 2 & 3 \\ x & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} z & 3 \\ 4 & 2 \end{bmatrix}$$

then $z=2$ and $x=4$

Rank of a Matrix $\rightarrow$ rank of a matrix gives the information of the solvency of the system.

following statements should be obeyed $\rightarrow$

- i) There is at least one minor of 'A' of order ' r ' which does not vanish.
- ii) every minor of A of order higher than r vanish.

Points to remember $\rightarrow$

- If matrix is zero matrix then its rank will be zero.
- To find rank of any matrix, the matrix must be a square matrix or if it is not then we should leave a column at the last.

Steps of finding rank $\rightarrow$ first we have to find out determinant of the matrix then if $\det \neq 0$ then rank will be (2) $\leftarrow$

(1) Coef. matrix $\rightarrow$ (2) find det $\rightarrow$ (3) check 0 or $\neq 0 \rightarrow$ (4) no. of rows which are non zero rows = rank $\rightarrow$ (5) if all rows becomes zero then rank will be 1.

Nullity: $\rightarrow$ To find nullity matrix should be a square matrix and denoted by $N(A)$.

$$N(A) = \text{order}(n) - r(A) [\text{rank}]$$

Ques: find rank of the following matrix

1	2	3	2
2	3	5	1
1	3	4	5

Sol: $\rightarrow$ Since this is not a square matrix then on leaving 1 column, we get.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

Now

$$|A| = 1(3 \times 4 - 3 \times 5) - 2(2 \times 4 - 5 \times 1) + 3(2 \times 3 - 3 \times 1) \\ = -3 - 6 + 9 \Rightarrow 0$$

then rank is $\neq 3$ of the matrix
Minors

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \Rightarrow -1$$

Hence rank = 2

$\rightarrow$ 2 Methods of finding rank of a matrix: $\rightarrow$

1 Determinant $\rightarrow$ It can be Zero or non-Zero.

Condition (1) $\rightarrow$ If det. is Zero then find all the possible minors.

If any minor is $\neq 0$ then no. of rows (non zero) will be the rank of the matrix. The matrix should be a square matrix. If the matrix is not a square matrix then leave 1 column from the matrix. If all the minors of order 2×2 then go for further solution and then the rank of matrix which you will get will be 1.

Echelon form: $\rightarrow$ Every upper triangular matrix is in the form of echelon form, but not the lower triangular matrix be in echelon form.

$\rightarrow$ Conditions for having an echelon form: $\rightarrow$
A matrix is called in echelon form if

- 1) The no. of zeros in the last row of the matrix should be more than the no. of zeros in its previous row of the matrix.
- 2) The no. of zeroes preceding the first non-zero element in a row is less than the no. of such zeroes in the next (or succeeding) row.
- 3) In the first row of matrix, the first entry should be non zero means it should be 1. In other words, the first non zero element in each row is unity (i.e. 1).

Point to remember: $\rightarrow$ If a matrix is in Echelon form, the rank of the matrix is equal to the no. of non zero rows in it.

Q4. find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Sol: $\rightarrow$ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

on applying $R_3 \rightarrow R_3 - 2R_1$ and $R_2 \rightarrow R_2 - R_1$ replacing

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in

triangular form then, which is in echelon form. No. of non zero rows are 2 then,

rank of matrix = no. of non zero rows
 $P(A) = 2$

→ Normal form or Canonical form: →

Matrix of the form: →

$$1) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I is the identity matrix of order r .

$$2) \begin{bmatrix} I_r & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$3) \begin{bmatrix} I_r \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$4) \begin{bmatrix} I_r \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ Where zero is the null matrix of any order.

Qy. If $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$ find the normal form of the matrix and also find its rank.

Sol: → as we know that

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

then on applying row operations as well as column operations, we get

$$R_1 \leftrightarrow R_3, R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow -R_2 \\ R_3 \rightarrow -R_3 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 6 & 2 & 4 \\ 0 & 1 & 5 & 10 \end{bmatrix}$$

$$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1, C_4 \rightarrow C_4 - 2C_1$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix}$$

$$R_2 \rightarrow -R_2, R_3 \rightarrow R_3 - 6R_2, C_3 \rightarrow C_3 - 5C_2$$

$$C_4 \rightarrow C_4 - 10C_2, R_3 \rightarrow \frac{R_3}{28}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C_4 \rightarrow C_4 - 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow [I_3 \ 0]$$

hence rank is 3. $\rho(A) = 3$.

Q4. If $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & 1 & -3 & 4 \end{bmatrix}$ then find normal form and rank.

Sol: $\rightarrow C_2 \leftrightarrow C_1, R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 8 & 3 & 6 \\ 0 & 0 & -7 & -16 \\ 0 & -16 & -6 & -2 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 8C_1, C_3 \rightarrow 3C_1 + C_3, C_4 \rightarrow C_4 - 6C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & -7 & -16 \\ 0 & -16 & -6 & -2 \end{bmatrix}$$

$C_2 \rightarrow -\frac{1}{8}C_2, R_3 \rightarrow R_3 - 2R_1, C_3 \rightarrow C_3 + 7C_2;$

$C_4 \rightarrow C_4 + 16C_2$ and

$R_3 \rightarrow \frac{1}{8}R_3$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 34/8 \end{bmatrix}$$

$$C_4 \rightarrow C_4 - \frac{34}{8} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow [I_3 \ 0]$$

Which is in normal form and hence rank is 3.

Solution Of A System of Linear Simultaneous Equation By Rank method

Consider the system of 3 Simultaneous equations in 3 variables x, y, z are given as

$$\left. \begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \right\} \rightarrow (1)$$

The system (1) can be written in the matrix form $AX = B \rightarrow (2)$

$$\text{Where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 34/8 \end{bmatrix}$$

$$C_4 \rightarrow C_4 - \frac{34}{8} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow [I_3 \ 0]$$

Which is in normal form and hence rank is 3.

Solution Of A System of Linear Simultaneous Equation By Rank Method

Consider the system of 3 simultaneous equations in 3 variables x, y, z are given as

$$\left. \begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned} \right\} \rightarrow (1)$$

The system (1) can be written in the matrix form $AX = B$ $\rightarrow (2)$

$$\text{Where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Condition for Consistency of System $Ax=B$

- (I) Augmented matrix $= [A : B]$
- (II) A Consistent system means Unique Solution or infinite Solution.
- (III) An inconsistent system means system have no solution.
- (IV) For the consistency of the system (2).
To find rank of $[A : B]$ and A ,
i.e. $\rho(A:B)$ and $\rho(A)$.

Case I: $\rightarrow$ If $\rho(A:B) = \rho(A) = r (= \text{number of variables } n)$

Then given system are consistent and they have Unique Solution.

Case II: $\rightarrow$ If $\rho(A:B) = \rho(A) = r (< \text{number of variables } n)$

Then given system are consistent and have infinite many solution. also $(n-r)$ variables will be linearly independent.

Case III: $\rightarrow$ If $\rho(A:B) \neq \rho(A)$, then system are inconsistent and they have no solution.

Q4 Show that the following equations are consistent and solve them.

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$x + 4y + 9z = 6$$

Sol: $\rightarrow$ Writing the system in matrix form $AX = B$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Now find the rank of Augmented matrix $[A:B]$ and A , we have

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 2 & 3 & : & 4 \\ 1 & 4 & 9 & : & 6 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 3 & 8 & : & 3 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - 3R_2$ we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 2 & : & 0 \end{bmatrix}$$

$\rightarrow (1)$

which is in echelon form

Clearly $\rho(A:B) = 3$ and $\rho(A) = 3$
 Hence $\rho(A:B) = \rho(A) = 3$ (= No. of Variables 3)

$\Rightarrow$ System is consistent and it has a Unique Solution.

rewrite eq (1) $2Z = 0$

$$Y + 2Z = 1$$

$$X + Y + Z = 3$$

Using back substitution $Z = 0, Y = 1, X = 2$

Solution is $x = 2, y = 1, z = 0$

~~Q4~~

Q4. prove that the equations

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$$2x - 3y - z = 5$$

are consistent and hence find the solution.

Sol: $\rightarrow$

$$AX = B$$

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array} \right]$$

operating $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, R_4 \rightarrow R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix}$$

$$R_3 \rightarrow 3R_3 + 10R_2, R_4 \rightarrow 3R_4 - 7R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 29 & : & -116 \\ 0 & 0 & -17 & : & 68 \end{bmatrix}$$

$$R_4 \rightarrow 29R_4 + 17R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 29 & : & -116 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \rightarrow \textcircled{1}$$

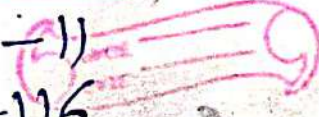
$$P(A:B) = 3 \text{ and } P(A) = 3$$

$P(A:B) = P(A) = 3$ (= No. of variables 3)
System is consistent and it has a
unique solution.

$$x + 2y = 4$$

$$-3y + 2z = -11$$

$$29z = -116$$



* Q4. Investigate the value of λ and μ so that the equations

$$2x + 3y + 5z = 9$$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

have i) no solution (ii) a unique solution
iii) an infinite number of solutions.

Sol: $\rightarrow$

$$AX = B$$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - (7/2)R_1$, $R_3 \rightarrow R_3 - R_1$

$$\sim = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -5/2 & -39/2 & : & -47/2 \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix} \rightarrow \textcircled{1}$$

Case I: $\rightarrow$ Given that eq (i) has no solution
When $\rho(A:B) \neq \rho(A)$.

Hence λ and μ must be $\lambda = 5$ & $\mu \neq 9$

Case II: $\rightarrow$ Given that system (i) has unique solution when

$\rho(A:B) = \rho(A) = 3$ (= No. of variables 3)

hence λ and μ must be

$\lambda \neq 5$ and μ may any value.

Case III: $\rightarrow$ Given that system (1) have infinite many solutions when
 $\rho(A:B) = \rho(A) = 2$ ($<$ no. of variables)
 hence λ and μ must be $\lambda = 5$ & $\mu = 9$.

Q4. Examine for consistency the following eqⁿ
 $5x + 3y + 14z = 4$, $y + 2z = 1$, $x - y + 2z = 0$
 and $2x + y + 6z = 2$.

Sol: $\rightarrow Ax = B$

$$[A:B] = \begin{bmatrix} 5 & 3 & 14 & : & 4 \\ 0 & 1 & 2 & : & 1 \\ 1 & -1 & 2 & : & 0 \\ 2 & 1 & 6 & : & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 5 & 3 & 14 & : & 4 \\ 2 & 1 & 6 & : & 2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_1$, $R_4 \rightarrow R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 8 & 4 & : & 4 \\ 0 & 3 & 2 & : & 2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 8R_2$, $R_4 \rightarrow R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & -4 & : & -1 \end{bmatrix}$$

$$R_4 \rightarrow 3R_4$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & -12 & : & -3 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

Which is in Echelon form

$$\rho(A:B) = 4 \text{ and } \rho(A) = 3$$

$$\rho(A:B) \neq \rho(A)$$

$\Rightarrow$ The system is inconsistent

$\Rightarrow$ System has no solution.

Solution of homogeneous Simultaneous equation

Consider :- $a_1x + b_1y + c_1z = 0$
 $a_2x + b_2y + c_2z = 0$
 $a_3x + b_3y + c_3z = 0$

Sol: Step 1 $\rightarrow$ write $Ax = B$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 2: $\rightarrow$ find rank of coefficient matrix by reducing it to echelon form.

Step 3: $\rightarrow$ on the basis of rank, we divide the solution according to following cases.

Case I: $\rho(A) = \text{no. of unknowns}$
System will have unique solution.
i.e. $x = y = z = 0$ (Trivial Solution)

Case II: $\rho(A) < \text{no. of unknowns}$
System will have infinite solution.
(Non trivial Solution)

Qy Solve the following system of equations

$$3x - y + z = 0$$

$$-15x + 6y - 5z = 0$$

$$5x - 2y + 2z = 0$$

sol: $\rightarrow$ Write $Ax = B$

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 5R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightleftharpoons R_3$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_1$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

hence rank of $A = 3 = \text{no. of unknowns}$
system will have unique solution
(Trivial solution) and i.e. $x=y=z=0$

Eigen Values and Eigen Vectors.

Characteristic equation $\Rightarrow$ Let

$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ be a square matrix and λ be an indeterminate then $|A - \lambda I| = 0$ is called characteristic equation of A .

for ex. $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

Eigen values or characteristic roots $\Rightarrow$

Let A be a square matrix of any order $|A - \lambda I| = 0$ it will give characteristic equation, on solving the values of λ are called eigen values or characteristic values or latent roots.

Qy. Determine the eigen values of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol: $\rightarrow |A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda-1)(\lambda-6) = 0$$

$$\Rightarrow \lambda = 1, 6 \text{ Ans.}$$

A

Qy. find the Eigen values of $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Sol: $\rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda) [(-2-\lambda)(3-\lambda) + 4] + 4 [3-\lambda-4] + 4 [-1+2+\lambda]$$

$$\Rightarrow (3-\lambda) [-6+2\lambda-3\lambda+\lambda^2+4] + 4(-\lambda-1) + 4(\lambda+1) = 0$$

$$\Rightarrow (3-\lambda) [\lambda^2 - \lambda - 2] - 4(\lambda+1) + 4(\lambda+1) = 0$$

$$\Rightarrow (3-\lambda)(\lambda+1)(\lambda-2) - 4(\lambda+1) + 4(\lambda+1) = 0$$

$$\Rightarrow (\lambda+1) [(3-\lambda)(\lambda-2) - 4 + 4] = 0$$

$$(\lambda+1)(\lambda-2)(3-\lambda) = 0$$

$$\lambda = -1, 2, 3 \text{ Ans.}$$

CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation.

Q4. Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and hence find } A^{-1}.$$

Sol: $\rightarrow$ The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$$
$$\Rightarrow -(1-\lambda^2) - 4 = 0$$
$$\lambda^2 - 5 = 0 \rightarrow \textcircled{1}$$

By Cayley-Hamilton theorem the matrix A must satisfy (1) i.e. putting $\lambda = A$ in (1), we get

$$A^2 - 5I = 0 \rightarrow \textcircled{2}$$

Verification: $\rightarrow$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

L.H.S. of (2) we have $A^2 - 5I$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

this verifies Cayley Hamilton theorem

To find A^{-1} : from (2) $A^2 - 5I = 0$
multiplying both sides by A^{-1} we get

$$A^{-1} A^2 - 5 A^{-1} I = A^{-1} 0 = 0$$

$$A^{-1} = \frac{1}{5} A = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \text{ Ans}$$

Q4. Verify Cayley-Hamilton theorem of the matrix $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1} .

Sol: $\rightarrow$ The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \rightarrow \textcircled{1}$$

By Cayley-Hamilton theorem the matrix A must satisfy (1) i.e put $\lambda = A$ in (1), we get

$$A^3 - 6A^2 + 9A - 4I = 0 \rightarrow (2)$$

Verification: $\rightarrow$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

hence $A^3 = A^2 \times A$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -22 & -2 \\ -2 & 22 & -2 \\ 2 & -2 & 22 \end{bmatrix}$$

Hence L.H.S. of (2), we have

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -22 & -2 \\ -2 & 22 & -2 \\ 22 & -2 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies Cayley-Hamilton theorem

To find A^{-1} : $\rightarrow$ Now multiplying both sides of (2) by A^{-1} we get

$$A^2 - 6A + 9I = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

Ans.

* Diagonalization Process

- (i) first we have to find Eigen Values
- (ii) for each eigen values we have to find Eigen vector
- (iii) we have to find matrix $P = [x_1 \ x_2 \ x_3]$
- (iv) we have to write diagonal matrix

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

for characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - |A| = 0$$

$S_1 =$ Sum of diagonal

$$S_1 = 1 + 2 + 3 = 6$$

$S_2 =$ Sum of the minor of diagonal

$$S_2 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix}$$

$$S_2 = 6 + 3 + (2 - 6) = 9 - 4 = 5$$

$$|A| = 1(6-0) - 6(3-0) + 1(0-0)$$

$$|A| = 6 - 18$$

$$|A| = -12$$

put the values in eq ①

$$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

for solve this eq we have find

$$\lambda = 4, 3, -1$$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} X = 0$$

$\lambda = 4$ for eigen vector

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-3x + 6y + z = 0$$

$$x = 2y$$

$$R_2 \leftrightarrow R_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 6 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y = 0$$

$$z = 0$$

$$\text{assume } y = 1$$

$$x - 2(1) = 0$$

$$\therefore x = 2$$

Eigen Vector

$$x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$

$$\begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + 6y + z = 0 \quad \text{--- (1)}$$

$$x - y = 0$$

$$x = y$$

$$\text{assume } y = 1$$

$$x = 1$$

put the value in eq (1)

$$-2(1) + 6(1) + z = 0$$

$$-2 + 6 + z = 0$$

$$4 + z = 0$$

$$\therefore z = -4$$

Eigen vector

$$x_2 =$$

$$\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

$$\underline{\underline{\lambda = -1}}$$

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_2 \leftrightarrow R_1$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \rightarrow R_3 - 4R_2$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + 3y = 0 \quad \text{--- (1)}$$

$$z = 0$$

assume

$$y = 1$$

put in eq (1)

$$x + 3(1) = 0$$

$$\therefore x = -3$$

Eigen Vector

$$x_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Now

$$\text{matrix } P = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

for diagonalization

$$P = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 0 & 3 \\ 0 & -4 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

→ Ans.

Unit - 2 $\rightarrow$ Calculus

$\rightarrow$ Limit of Sum $\checkmark$

$\rightarrow$ Sumation of Series $\checkmark$

$\rightarrow$ Beta and Gamma Function $\checkmark$

$\rightarrow$ Double and triple integrals $\checkmark$

$\rightarrow$ Application of integral

$\rightarrow$ Volume & Area by Revolution

* Summation of Series :-

Q:- Find the limit of the series when $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{n}{8n} \right]$$

$$\lim_{n \rightarrow \infty} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{n^2}{(n+n)^3} \right]$$

$$\lim_{n \rightarrow \infty} \sum_{r=0}^{n \leq n} \frac{n^2}{(n+r)^3}$$

$$\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{n^2}{n^3 \left(1 + \frac{r}{n}\right)^3}$$

$$\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{\left(1 + \frac{r}{n}\right)^3} \cdot \frac{1}{n}$$

$$\rightarrow \int_0^1 \frac{1}{(1+x)^3} dx$$

$$\Rightarrow \int_0^1 (1+x)^{-3} dx$$

$$\Rightarrow \left[\frac{(1+x)^{-3+1}}{-3+1} \right]_0^1$$

$$\rightarrow \left[\frac{(1+x)^{-2}}{-2} \right]_0^1$$

$$\Rightarrow \frac{1}{-2} \left[\frac{1}{(1+x)^2} \right]_0^1 \Rightarrow \frac{1}{-2} \left[\frac{1}{4} - \frac{1}{1} \right]$$

$$\Rightarrow \frac{-\frac{1}{2} \times -\frac{3}{4}}{1} = \frac{3}{8} \quad \underline{\text{Ans}}$$

Q:- Evaluate

$$\lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}$$

assume

$$A = \lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}$$

$$\log A = \lim_{n \rightarrow \infty} \log \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}$$

$$\log A = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}$$

$$\log A = \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \log \left(1 + \frac{1}{n}\right) + \log \left(1 + \frac{2}{n}\right) \dots \log \left(1 + \frac{n}{n}\right) \right\}$$

$$\log A = \frac{1}{n} \lim_{n \rightarrow \infty} \sum_{k=1}^n \log \left(1 + \frac{x_k}{n} \right)$$

replace $\frac{1}{n} = dx$, $\lim_{n \rightarrow \infty} \sum_{k=1}^n = \int_0^1$
 $\frac{x_k}{n} = x$

$$\log A = \int_0^1 \log(1+x) \cdot dx$$

$$\log A = \int \log(1+x) \cdot dx - \int \left(\frac{d}{dx} \log(1+x) \int dx \right) \cdot dx$$

$$\log A = \left[\log(1+x) \cdot x - \int \frac{1}{1+x} \cdot x \cdot dx \right]_0^1$$

$$\log A = x \log(1+x) - \int \frac{x+1-1}{1+x} dx$$

$$\log A = x \log(1+x) - \left[\int dx - \int \frac{1}{1+x} dx \right]$$

$$\log A = \left[x \log(1+x) - [x - \log(1+x)] \right]_0^1$$

$$\log A = \log(2) - [1 - \log(2)]$$

$$\log A = 2 \log(2) - 1$$

$$\log A = \log(4) - \log e$$

$$\log A = \log\left(\frac{4}{e}\right)$$

$$A = \frac{4}{e}$$

Ans.

Q:- Evaluate

$$\lim_{n \rightarrow \infty} \left\{ \frac{n!}{n^n} \right\}^{1/n}$$

Soln:- $\lim_{n \rightarrow \infty} \left\{ \frac{1 \times 2 \times 3 \times \dots \times n}{n \times n \times n \times \dots \times n} \right\}^{1/n}$

$$\lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right\}^{1/n}$$

$$A = \lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right\}^{1/n}$$

$$\log A = \lim_{n \rightarrow \infty} \log \left\{ \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right\}^{1/n}$$

$$\log A = \frac{1}{n} \lim_{n \rightarrow \infty} \log \left\{ \left(\frac{1}{n} \right) \left(\frac{2}{n} \right) \left(\frac{3}{n} \right) \dots \left(\frac{n}{n} \right) \right\}$$

$$\log A = \frac{1}{n} \lim_{n \rightarrow \infty} \left\{ \log \left(\frac{1}{n} \right) + \log \left(\frac{2}{n} \right) + \dots + \log \left(\frac{n}{n} \right) \right\}$$

$$\log A = \frac{1}{n} \sum_{k=1}^n \log \left(\frac{k}{n} \right)$$

Replace

$$\log A = \int_0^1 \log(x) dx$$

$$\log A = \left[x \log(x) - x \right]_0^1$$

$$\log A = \log(1) - 1$$

$$\log A = -1$$

Ans.

Beta and Gamma function:-

* Beta function:-

- It is defined by $\beta(m, n)$
- defined as definite integral

$$\beta(m, n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$$

Properties of Beta function

$$(1) \quad \beta(m, n) = \beta(n, m)$$

$$(2) \quad \beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} \cdot dx$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \cdot dx$$

$$3) \beta(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta$$

Gamma Function

→ It is denoted by Γ

→ Definite integral

$$\Gamma = \int_0^{\infty} e^{-x} \cdot x^{n-1} dx$$

($n > 0$)

Properties

① $\Gamma 1 = 1$

② $\Gamma(n+1) = n!$

③ $\Gamma n = z^n \int_0^{\infty} e^{-zx} x^{n-1} dx$

④ $\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$

⑤ $\Gamma 1/2 = \sqrt{\pi}$

* Relation between Beta and Gamma
Prove that

$$\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} \quad (m, n > 0)$$

property

Soln:-

$$\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} dx$$

$$\Gamma(n) = z^n \int_0^{\infty} e^{-zx} \cdot x^{n-1} dx \quad \text{--- (1)}$$

$$\frac{\Gamma(n)}{z^n} = \int_0^{\infty} e^{-zx} \cdot x^{n-1} dx \quad \text{--- (II)}$$

In eq (1) multiply both side $\int_0^{\infty} e^{-z} \cdot z^{m-1} dz$

$$\Gamma(n) \int_0^{\infty} e^{-z} \cdot z^{m-1} dz = \int_0^{\infty} e^{-zx} \cdot x^{n-1} dx \left[\int_0^{\infty} e^{-z} \cdot z^{m-1} dz \right]$$

$$\left\{ \begin{array}{l} \because \Gamma(n) = \int_0^{\infty} e^{-z} \cdot z^{n-1} dz \\ \therefore \Gamma(m) = \int_0^{\infty} e^{-z} \cdot z^{m-1} dz \end{array} \right.$$

$$\Rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} \int_0^{\infty} z^n e^{-zx} \cdot x^{n-1} \cdot e^{-z} \cdot z^{m-1} dx dz$$

$$\Rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} x^{n-1} dx \int_0^{\infty} z^n e^{-zx} \cdot e^{-z} \cdot z^{m-1} dz$$

$$\Rightarrow \Gamma(n) \Gamma(m) = \int_0^{\infty} x^{n-1} dx \int_0^{\infty} e^{-zx-x} \cdot z^{m-1+n} dz$$

Prove that

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Q.

$$\Gamma(1/2) = \sqrt{\pi}$$

Soln:-

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\beta(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} \cdot (\cos \theta)^{2n-1} \cdot d\theta$$

$$\frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} \cdot (\cos \theta)^{2n-1} d\theta$$

but $m = 1/2$ and $n = 1/2$ both side

$$\frac{\Gamma(1/2) \Gamma(1/2)}{\Gamma(1/2 + 1/2)} = 2 \int_0^{\pi/2} (\sin \theta)^{1-1} \cdot (\cos \theta)^{1-1} d\theta$$

$$\frac{(\Gamma(1/2))^2}{\Gamma(1)} = 2 \int_0^{\pi/2} d\theta$$

$$\left\{ \because \Gamma(1) = 1 \right\}$$

$$(\Gamma(1/2))^2 = 2 \left[\theta \right]_0^{\pi/2}$$

$$(\Gamma(1/2))^2 = 2 \cdot \frac{\pi}{2}$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Hence proved

Q:- Evaluate

$$\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}} dx$$

Soln:- $\Rightarrow \int_0^{\infty} \frac{x^8 - x^{14}}{(1+x)^{24}} \cdot dx$

$$\Rightarrow \int_0^{\infty} \frac{x^8}{(1+x)^{24}} - \frac{x^{14}}{(1+x)^{24}} \cdot dx$$

$$\Rightarrow \int_0^{\infty} \frac{x^8}{(1+x)^{24}} \cdot dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}} \cdot dx$$

$$\Rightarrow \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} \cdot dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}} \cdot dx$$

$$\left[\because \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m, n) \right]$$

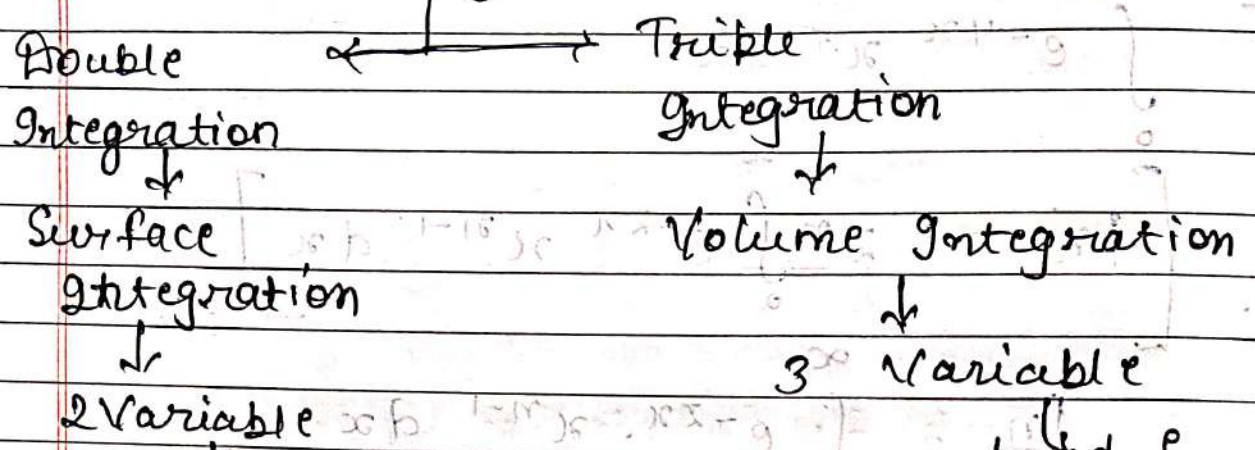
$$\Rightarrow \beta(9, 15) - \beta(15, 9)$$

$$\left\{ \beta(n, m) = \beta(m, n) \right\}$$

$$\Rightarrow \beta(15, 9) - \beta(15, 9) = 0$$

Ans

Multiple Integration



$$\int_a^b \int_c^d f(x, y) dx dy$$

$$\int_a^b \int_c^d \int_e^f f(x, y, z) dx dy dz$$

* Double Integration

Constant

$$\int_1^3 \int_2^4 (x+y)^2 dx dy$$

* this is a limit of x

this is y limit of y

* limit Variable

$$\int_0^x x^2 = y \cdot dy$$

$$\int_{y^2}^y y^2 = x dx$$

$$\int x^2 y^2 z^2 dz$$

$$x^2 y^2 \int z^2 dz$$

$$x^2 y^2 \cdot \frac{z^{2+1}}{2+1} + C$$

$$x^2 y^2 \frac{z^3}{3} + C$$

Ans.

$$\int \sin(2x+9) \cdot dx$$

$$-\frac{\cos(2x+9)}{2} + C$$

Ans.

★★

$$\int \frac{f'(x)}{f(x)} \cdot dx = \log |f(x)| + C$$

$$\int \frac{2x}{x^2+1} \cdot dx = \log(x^2+1) + C$$

$$\int \frac{3x^2}{x^3+9} \cdot dx = \log(x^3+9) + C$$

$$\int \frac{x}{x^2+5} \cdot dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2x}{x^2+5} \cdot dx = \frac{1}{2} \log(x^2+5) + C$$

Ans

$$\underline{\text{Q.1}} \int_0^2 \int_0^1 (x^2 + y^2) dx dy$$

$$\Rightarrow \int_0^2 dy \int_0^1 (x^2 + y^2) \cdot dx$$

$$\Rightarrow \int_0^2 dy \int_0^1 x^2 \cdot dx + y^2 \cdot dx$$

$$\Rightarrow \int_0^2 dy \left[\int_0^1 x^2 \cdot dx + \int_0^1 y^2 \cdot dx \right]$$

$$\Rightarrow \int_0^2 dy \left[\frac{x^3}{3} + y^2 x \right]_0^1$$

$$\Rightarrow \int_0^2 dy \left[\frac{1}{3} + y^2 \right]$$

$$\Rightarrow \frac{1}{3} \int_0^2 dy + \int_0^2 y^2 \cdot dy$$

$$\Rightarrow \frac{1}{3} \left[\frac{y}{1} + \frac{y^3}{3} \right]_0^2$$

$$\Rightarrow \left[\frac{2}{3} + \frac{(2)^3}{3} \right] \Rightarrow \frac{2}{3} + \frac{8}{3} = \frac{10}{3}$$

Ans.

Q:- $\int_1^2 \int_1^3 x y^2 dx dy$

$$\rightarrow \int_1^2 dy \int_1^3 x y^2 \cdot dx$$

$$\rightarrow \int_1^2 y^2 \cdot dy \int_1^3 x \cdot dx$$

$$\rightarrow \int_1^2 y^2 \cdot dy \left[\frac{x^2}{2} \right]_1^3$$

$$\rightarrow \int_1^2 y^2 \cdot dy \left[\frac{(3)^2}{2} - \frac{(1)^2}{2} \right]$$

$$\rightarrow \int_1^2 y^2 \cdot dy \left[\frac{9}{2} - \frac{1}{2} \right]$$

$$\rightarrow 4 \int_1^2 y^2 \cdot dy$$

$$\rightarrow 4 \left[\frac{y^3}{3} \right]_1^2$$

$$\rightarrow 4 \left[\frac{(2)^3}{3} - \frac{(1)^3}{3} \right]$$

$$\Rightarrow 4 \left[\frac{8}{3} - \frac{1}{3} \right] \Rightarrow \frac{4 \times 7}{3} = \frac{28}{3}$$

Ans

Q:- $\int_0^1 \int_0^{x^2} e^{y/x} \cdot dy \cdot dx$

Soln:- $\Rightarrow \int_0^1 dx \int_0^{x^2} e^{y/x} \cdot dy$

$$\Rightarrow \int_0^1 dx \left[\frac{e^{y/x}}{\frac{1}{x}} \right]_0^{x^2}$$

$$\Rightarrow \int_0^1 dx \left[x e^{y/x} \right]_0^{x^2}$$

$$\Rightarrow \int_0^1 dx \left[x \cdot e^{x^2/x} \right]$$

$$\Rightarrow \int_0^1 dx \left[x \cdot e^x \right] - x$$

$$\Rightarrow \int_0^1 x \cdot e^x - x dx$$

$$\Rightarrow \int_I x \cdot \frac{e^x}{II}$$

$$\Rightarrow x \int e^x - \int \left(\frac{dx}{dx} \int e^x \cdot dx \right) dx$$

$$\Rightarrow x e^x - e^x + C$$

put the value in eq ①

$$\Rightarrow \int_0^1 \left[x e^x - e^x \frac{x}{2} \right]_0$$

$$\Rightarrow \left\{ \left(e^1 - \frac{e^1}{2} \right) - \left(0 \times e^0 - \frac{e^0}{2} \right) \right\}$$

$$\Rightarrow \frac{-0 + e^0}{2} = \frac{1 - \frac{1}{2}}{2} = \frac{1}{2}$$

Ans

Qr

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx \cdot dy}{1+x^2+y^2}$$

Soln

$$\int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{dy}{(\sqrt{1+x^2})^2 + y^2}$$

$$\Rightarrow \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c \right]$$

$$\Rightarrow \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1} \left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}}$$

$$\Rightarrow \int_0^1 dx \cdot \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \right) \right]$$

$$\rightarrow \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot dx$$

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$$\left[\int \frac{dx}{\sqrt{1+x^2}} = \log(x + \sqrt{1+x^2}) + C \right]$$

$$\rightarrow \frac{\pi}{4} \left[\log(x + \sqrt{1+x^2}) \right]_0^1$$

$$\rightarrow \frac{\pi}{4} \left[\log(1 + \sqrt{1+1}) \right]$$

$$\rightarrow \frac{\pi}{4} \log(1 + \sqrt{2})$$

Ans.

* Triple Integration

$$\int_a^b \int_c^d \int_e^f f(x, y, z) dx dy dz$$

Constant

Variable

$$\int_0^1 \int_1^2 \int_2^3 xyz dx dy dz$$

Diagram showing the order of integration: x is the innermost integral (labeled 1), y is the middle integral (labeled 2), and z is the outermost integral (labeled 3). Arrows point from the labels to the corresponding integrals.

$$\int_1^2 \int_2^4 \int_4^6 xyz dy dz dx$$

Diagram showing the order of integration: y is the innermost integral (labeled 1), z is the middle integral (labeled 2), and x is the outermost integral (labeled 3). Arrows point from the labels to the corresponding integrals.

①

$$\int_0^2 \int_0^{x^2} \int_{\frac{x+y}{x}}^{x-y} xyz dx dy dz$$

$$\int_{x=0}^2 x dx \int_{y=0}^{x^2} y dy \int_{z=x+y}^{x-y} z dz$$

Q:-
$$\int_0^3 \int_0^2 \int_0^1 (x+y+z) dx dy dz$$

Soln:-
$$\rightarrow \int_0^3 dx \int_0^2 dy \int_0^1 (x+y+z) dz$$

$$\rightarrow \int_0^3 dx \int_0^2 dy \int_0^1 x dz + y dz + z dz$$

$$\rightarrow \int_0^3 dx \int_0^2 dy \left[xz + yz + \frac{z^2}{2} \right]_0^1$$

$$\rightarrow \int_0^3 dx \int_0^2 dy \left[x + y + \frac{1}{2} \right]$$

$$\rightarrow \int_0^3 dx \int_0^2 x \cdot dy + y \cdot dy + \frac{1}{2} dy$$

$$\rightarrow \int_0^3 dx \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_0^2$$

$$\rightarrow \int_0^3 dx [2x + 2 + 1]$$

$$\rightarrow \int_0^3 2x dx + 2 dx + dx$$

$$\rightarrow \int_0^3 2x dx + 2 dx + dx$$

$$\rightarrow \left[\frac{2x^2}{2} + 2x + x \right]_0^3$$

$$\rightarrow [(3)^2 + 2(3) + 3] = 9 + 6 + 3 = 18$$

Ans.

Q. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

Soln: $\int_0^a \int_0^x \int_0^{x+y} e^x \cdot e^y \cdot e^z \cdot dz dy dx$

$$\rightarrow \int_0^a e^x \cdot dx \int_0^x e^y \cdot dy \int_0^{x+y} e^z \cdot dz$$

$$\rightarrow \int_0^a e^x \cdot dx \int_0^x e^y \cdot dy \cdot [e^z]_0^{x+y}$$

$$\rightarrow \int_0^a e^x dx \int_0^x e^y dy [e^{x+y} - e^0]$$

$$\Rightarrow \int_0^a e^x dx \int_0^x e^y dy [e^x \cdot e^y - 1]$$

$$\Rightarrow \int_0^a e^x dx \int_0^x e^y \cdot e^x \cdot e^y \cdot dy - e^y \cdot dy$$

$$\Rightarrow \int_0^a e^x dx \int_0^x e^{2y} \cdot e^x \cdot dy - e^y dy$$

$$\Rightarrow \int_0^a e^x dx \left[\frac{e^{2y} \cdot e^x}{2} - e^y \right]_0^x$$

$$\Rightarrow \int_0^a e^x dx \left\{ \left[\frac{e^{2x} \cdot e^x}{2} - e^x \right] - \left[\frac{e^0 \cdot e^x}{2} - e^0 \right] \right\}$$

$$\Rightarrow \int_0^a e^x dx \left\{ \left[\frac{e^{3x}}{2} - e^x - \frac{e^x}{2} + 1 \right] \right\}$$

$$\Rightarrow \int_0^a \left\{ \frac{e^{4x}}{2} dx - e^{2x} dx - \frac{e^{2x}}{2} + e^x \right\} dx$$

$$\Rightarrow \int_0^a \frac{e^{4x}}{2} dx - e^{2x} dx - \frac{e^{2x}}{2} dx + e^x dx$$

$$\Rightarrow \left[\frac{e^{4x}}{4 \times 2} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x \right]_0^a$$

$$\Rightarrow \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^x \right]_0^a$$

$$\Rightarrow \left[\frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^a \right] - \left[\frac{e^0}{8} - \frac{e^0}{2} - \frac{e^0}{4} + e^0 \right]$$

$$\Rightarrow \left[\frac{e^{4a} - 4e^{2a} - 2e^{2a} + 8e^a}{8} \right] - \left[\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1 \right]$$

$$\Rightarrow \left[\frac{e^{4a} - 6e^{2a} + 8e^a}{8} \right] - \left[\frac{1 - 4 - 2 + 8}{8} \right]$$

$$\Rightarrow \left[\frac{e^{4a} - 6e^{2a} + 8e^a}{8} \right] - \left[\frac{3}{8} \right]$$

$$\Rightarrow \frac{e^{4a} - 6e^{2a} + 8e^a - 3}{8}$$

Ans.

Q.: Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$$

Soln:

$$\int_0^1 x \, dx \int_0^{\sqrt{1-x^2}} y \, dy \int_0^{\sqrt{1-x^2-y^2}} z \, dz$$

$$\Rightarrow \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y dy \int_0^{\sqrt{1-x^2-y^2}} z \cdot dz$$

$$\Rightarrow \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y dy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y dy \left[\frac{(\sqrt{1-x^2-y^2})^2}{2} \right]$$

$$\Rightarrow \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y dy \left(\frac{1-x^2-y^2}{2} \right)$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \int_0^{\sqrt{1-x^2}} y - yx^2 - y^3 dy$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \left[\frac{y^2}{2} - \frac{yx^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{1-x^2}}$$

$$\int_0^1 \frac{x}{2} dx \left[\frac{y^2}{2} \left\{ 1 - x^2 - \frac{y^2}{2} \right\} \right]_0^{\sqrt{1-x^2}}$$

$$\int_0^1 \frac{x}{2} dx \left[\frac{(\sqrt{1-x^2})^2}{2} \left\{ 1 - x^2 - \frac{(\sqrt{1-x^2})^2}{2} \right\} \right] - \left[0 \right]$$

$$\int_0^1 \frac{x}{2} dx \left[\frac{1-x^2}{2} \left\{ 1 - x^2 - \frac{(1-x^2)}{2} \right\} \right]$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \left\{ \frac{y^2}{2} (1-x^2) - \frac{y^4}{4} \right\} \sqrt{1-x^2}$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \left\{ \frac{(\sqrt{1-x^2})^2 (1-x^2)}{2} - \frac{(\sqrt{1-x^2})^4}{4} \right\}$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \left\{ \frac{(1-x^2)}{2} \cdot (1-x^2) - \frac{(1-x^2)^2}{4} \right\}$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx \left\{ \frac{(1-x^2)^2}{2} - \frac{(1-x^2)^2}{4} \right\}$$

$$\Rightarrow \int_0^1 \frac{x}{2} \cdot dx \left\{ \frac{(1-x^2)^2}{4} \right\} \cdot du$$

$$\Rightarrow \frac{1}{8} \int_0^1 x(1-x^2)^2$$

$$\Rightarrow \frac{1}{8} \int_0^1 x(1+x^4-2x^2) \cdot dx$$

$$\Rightarrow \frac{1}{8} \int_0^1 x dx + x^5 \cdot dx - 2x^3 \cdot dx$$

$$\Rightarrow \frac{1}{8} \left[\frac{x^2}{2} + \frac{x^6}{6} - 2 \cdot \frac{x^4}{4} \right]_0^1$$

$$\Rightarrow \frac{1}{8} \left[\frac{1}{2} + \frac{1}{6} - \frac{1}{2} \right] = \frac{\frac{3+1-3}{6}}{8} = \frac{1}{48}$$

Q:-
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx dy dz}{\sqrt{1-x^2-y^2-z^2}}$$

Soln:-
$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}}$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(\sqrt{1-x^2-y^2})^2 - (z)^2}}$$

$$\Rightarrow \left[\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C \right]$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \sin^{-1} \left(\frac{z}{\sqrt{1-x^2-y^2}} \right) \cdot dz$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \left[\sin^{-1} \left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} \right) \right]$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) \right] dy$$

$$\Rightarrow \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} \cdot dy$$

$$\Rightarrow \frac{\pi}{2} \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy$$

$$\rightarrow \frac{\pi}{2} \int_0^1 dx [y]_0^{\sqrt{1-x^2}}$$

$$\rightarrow \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} \cdot dx$$

$$\left[\because \int \sqrt{1-x^2} dx = \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x + C \right]$$

$$\rightarrow \frac{\pi}{2} \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1$$

$$\rightarrow \frac{\pi}{2} \left[\frac{1\sqrt{1-1^2}}{2} + \frac{1}{2} \sin^{-1}(1) \right]$$

$$\rightarrow \frac{\pi}{2} \left[0 + \frac{1}{2} \times \frac{\pi}{2} \right] \Rightarrow \frac{\pi^2}{8}$$

Ans

$$\underline{\underline{Q.}} \int_0^1 \int_0^z \int_0^{x+z} (x+y+z) dx dy dz$$

Soln:-

$$\int_0^1 dz \int_0^z dx \int_0^{x+z} (x+y+z) dy$$

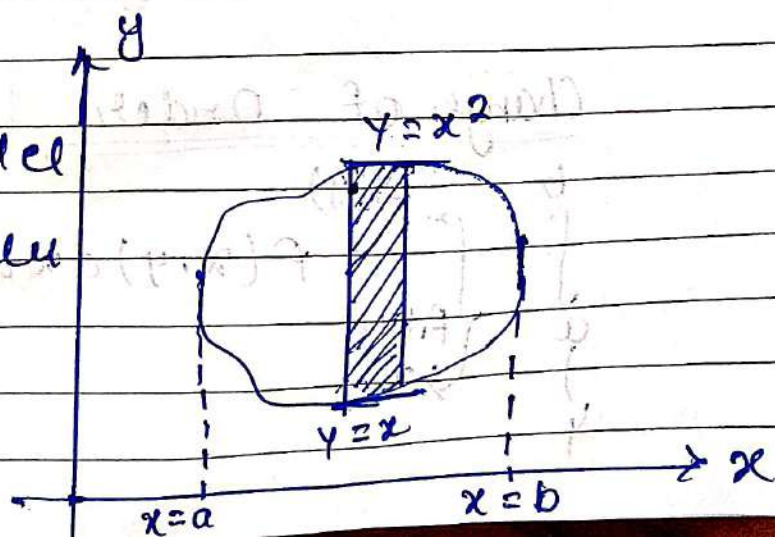
$$\rightarrow \int_0^1 dz \int_0^z dx \int_0^{x+z} x dy + y dy + z dy$$

Change of Order

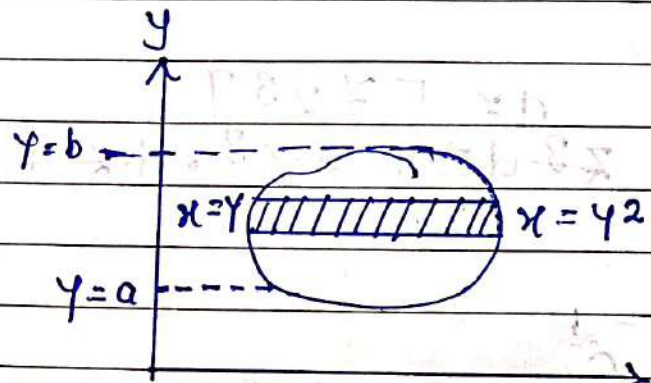
↳ Change Double Integration

$$\int_a^b \int_x^{x^2} f(x, y) dx dy$$

$\left\{ \begin{array}{l} y \text{ limit} \rightarrow y \text{ axis parallel} \\ x \text{ limit} \rightarrow x \text{ axis parallel} \end{array} \right.$



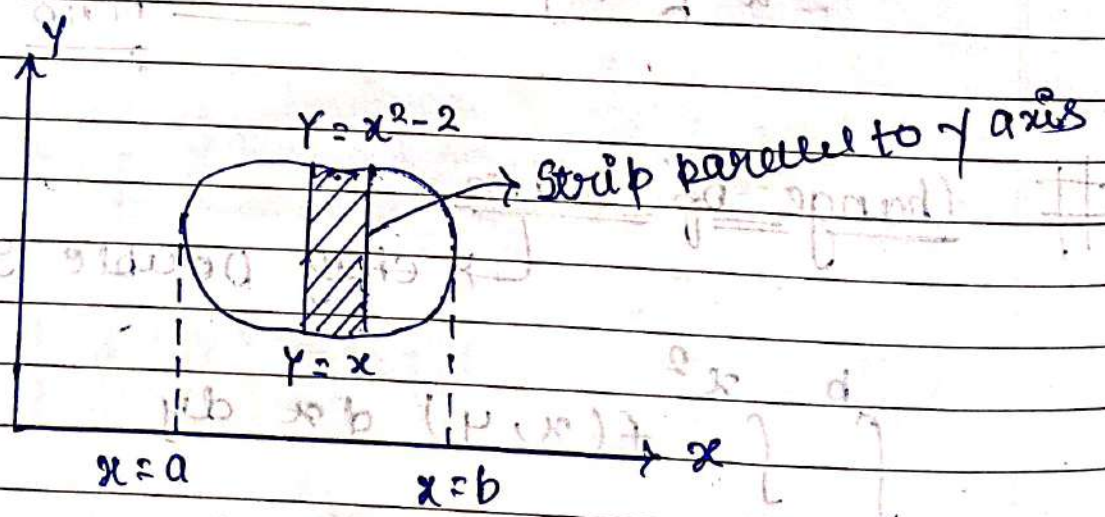
*
$$\int_a^b \int_y^{y^2} f(x, y) dx dy$$



Q:-
$$\int_a^b \int_{x^2-2}^{x^2-2} f(x, y) dx dy$$

$x \rightarrow$ Variable $\rightarrow$ (strip - parallel to y axis)
 $y =$ limit

$x=a$ & $x=b$



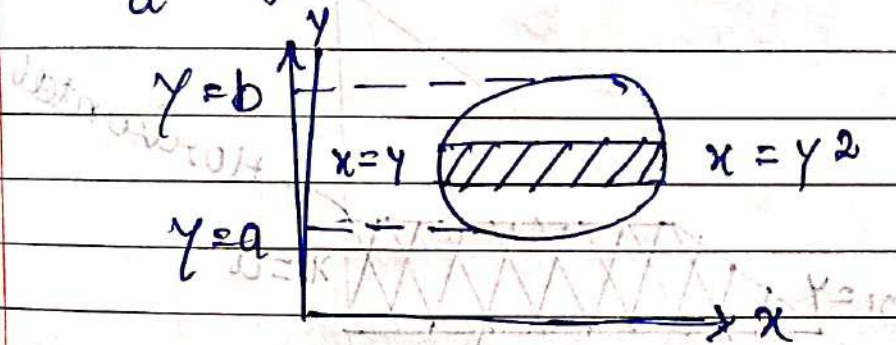
Change of Order

$$\int_a^b \int_{f_1(y)}^{f_2(y)} f(x, y) dx dy = \int_c^d \int_{F(x)}^{G(x)} f(x, y) dy dx$$

Ex:- $\int_0^1 \int_{x^2}^{x^2-1} f(x,y) dy dx \rightarrow \int_0^2 \int_{y^2}^{y^2-1} f(x,y) dx dy$

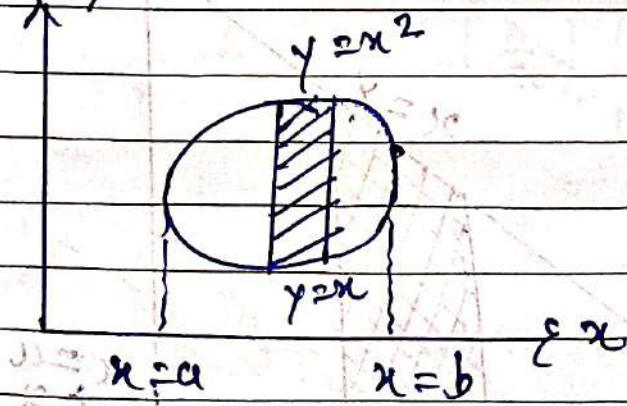
Diagram showing the region of integration in the first quadrant, bounded by $y = x^2$ and $y = x^2 - 1$. The region is shaded in pink.

* $\int_a^b \int_y^{y^2} f(x,y) dx dy$



After change of order

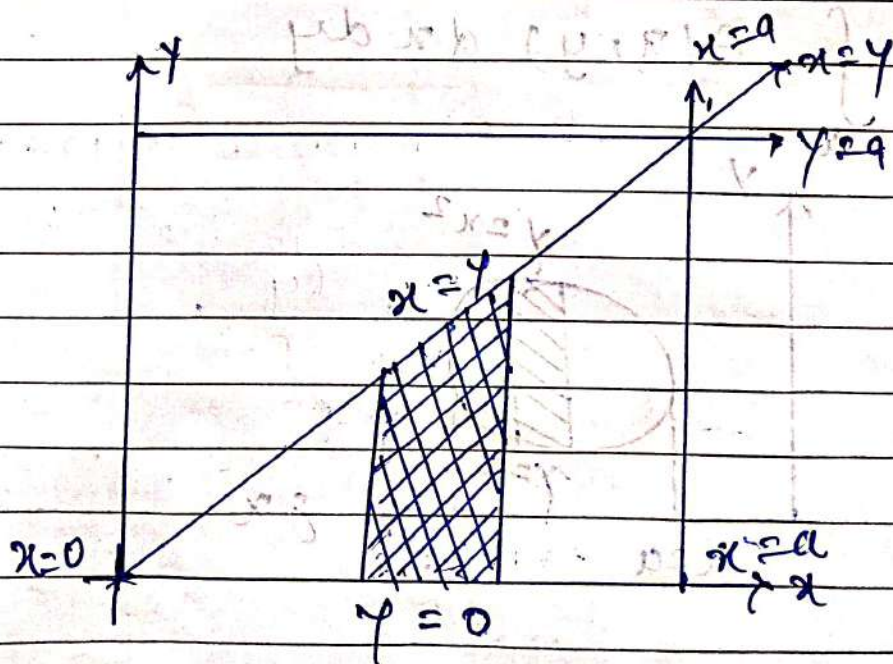
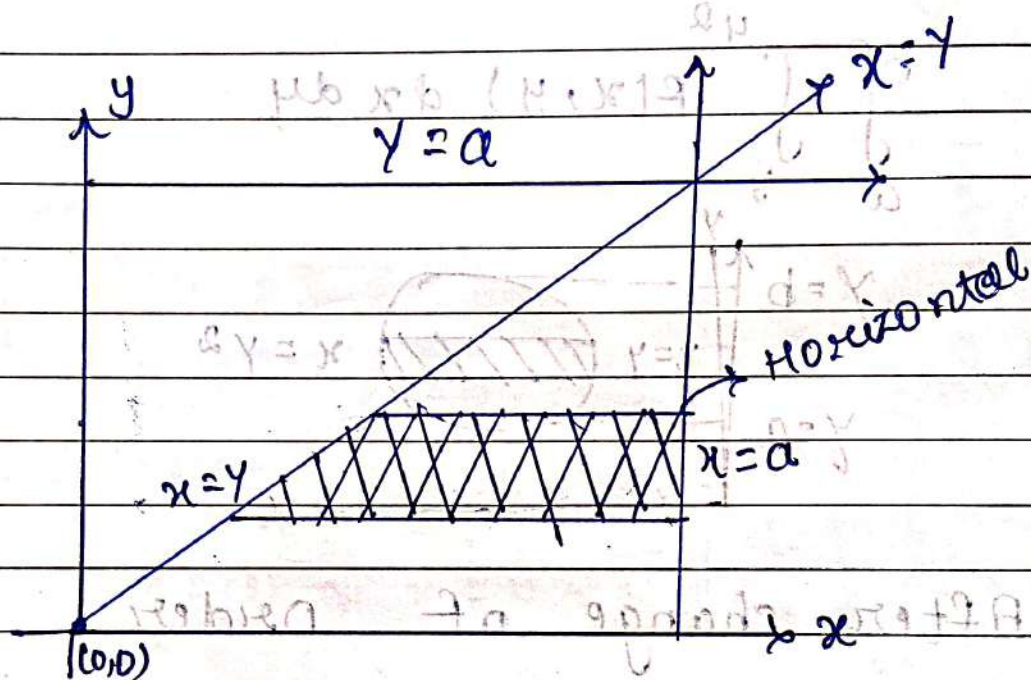
$\int_a^b \int_{x^2}^{x^2} f(x,y) dx dy$



Q

$$\int_0^a \int_y^a \frac{x}{x^2 + y^2} dx dy$$

$$\begin{aligned} x &= y \text{ to } x = a \\ y &= 0 \text{ to } y = a \end{aligned}$$



After change of order

$$x=0, \quad x=a$$

$$y=0, \quad \text{or } y=x$$

$$\Rightarrow \int_{x=0}^{x=a} \int_{y=0}^{y=x} \frac{x}{x^2+y^2} \cdot dx \, dy$$

$$\Rightarrow \int_{x=0}^{x=a} x \, dx \int_{y=0}^{y=x} \frac{1}{x^2+y^2} \cdot dy$$

$$\Rightarrow \int \int \frac{1}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow \int_{x=0}^{x=a} x \, dx \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x$$

$$\Rightarrow \int x \, dx \left[\frac{1}{x} \tan^{-1}\left(\frac{x}{x}\right) \right]$$

$$\Rightarrow \int_0^a x \, dx \left[\frac{1}{x} \cdot \frac{\pi}{4} \right]$$

$$\Rightarrow \frac{\pi}{4} \int_0^a dx$$

$$\Rightarrow \frac{\pi}{4} [x]_0^a$$

$$\Rightarrow \frac{\pi}{4} \times a$$

Ans.

Application of integration

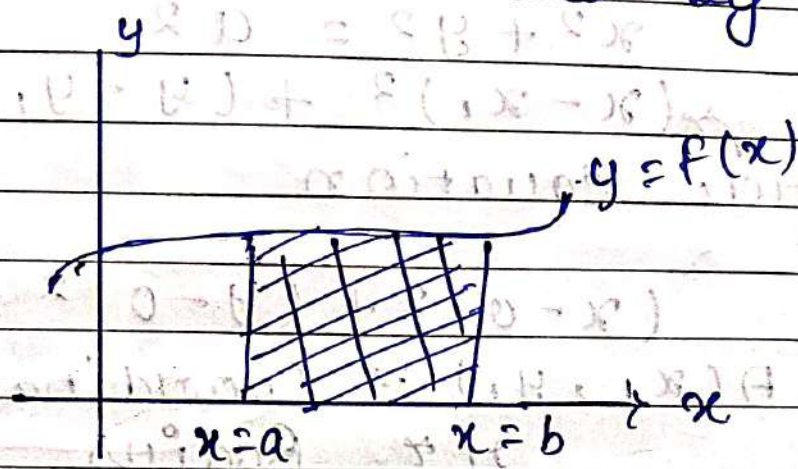
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① Area

Case-I

Area Bounded by curve

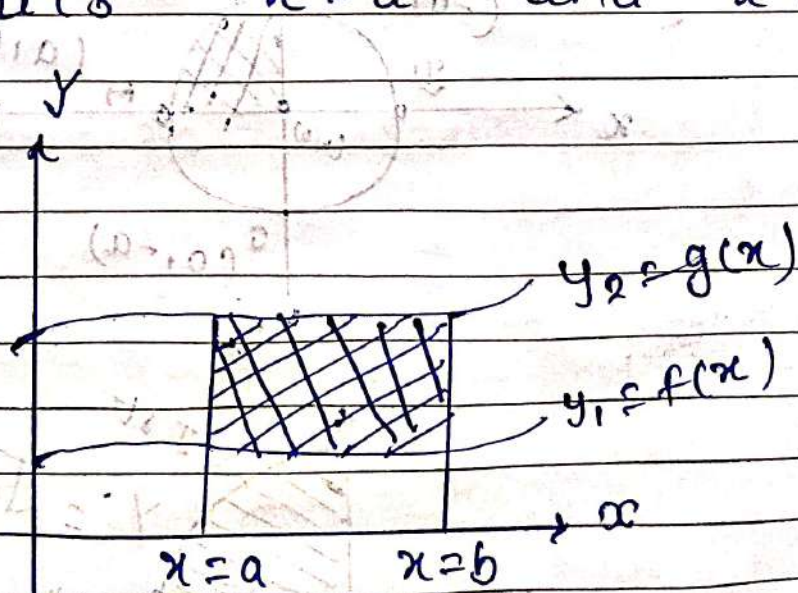


$$\text{Area} = \int_a^b f(x) dx$$

Case II

Area bounded by two curves

$y_1 = f(x)$, $y_2 = g(x)$ between the limits $x = a$ and $x = b$



$$\text{Area} = \int_a^b g(x) dx - \int_a^b f(x) dx$$

Q:- Find the whole area of circle $x^2 + y^2 = a^2$ by definite integral

Soln:-

$$x^2 + y^2 = a^2$$

$$(x - x_1)^2 + (y - y_1)^2 = a^2$$

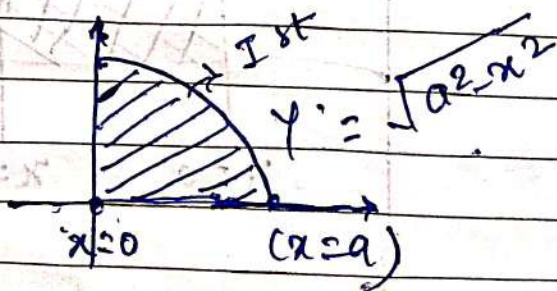
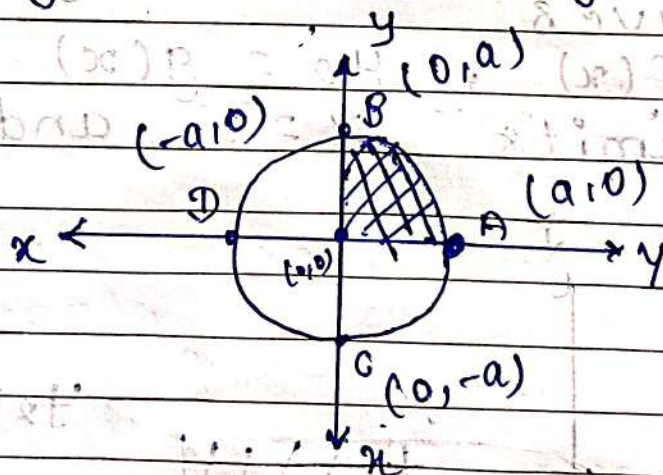
Circle Equation

$$(x - 0)^2 + (y - 0)^2 = a^2$$

$A(x_1, y_1)$ * Coordinate of circle
* Radius

Center $(x_1, y_1) \rightarrow (0, 0) \rightarrow$ origin
radius $= r = a$

by definite integral



$$x^2 + y^2 = a^2$$

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$

Total Area of $\square$

$$\Rightarrow 4 \int_0^a \sqrt{a^2 - x^2} \cdot dx$$

$$\Rightarrow \int_0^a \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow 4 \int_0^a \sqrt{a^2 - x^2} \cdot dx$$

$$\Rightarrow 4 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$\Rightarrow 4 \left[\frac{a^2}{2} \sin^{-1}\left(\frac{a}{a}\right) \right]$$

$$\Rightarrow 4 \cdot \frac{a^2}{2} \sin^{-1}(1)$$

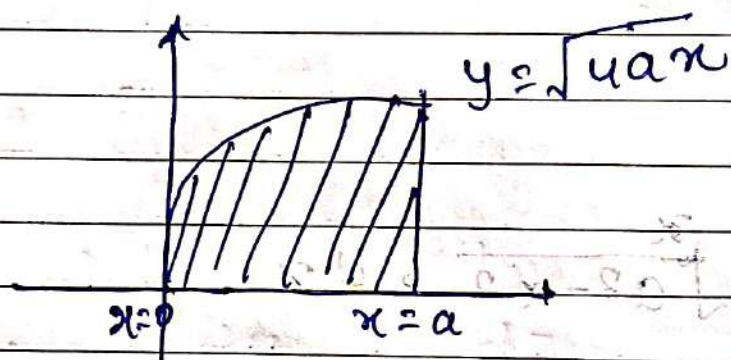
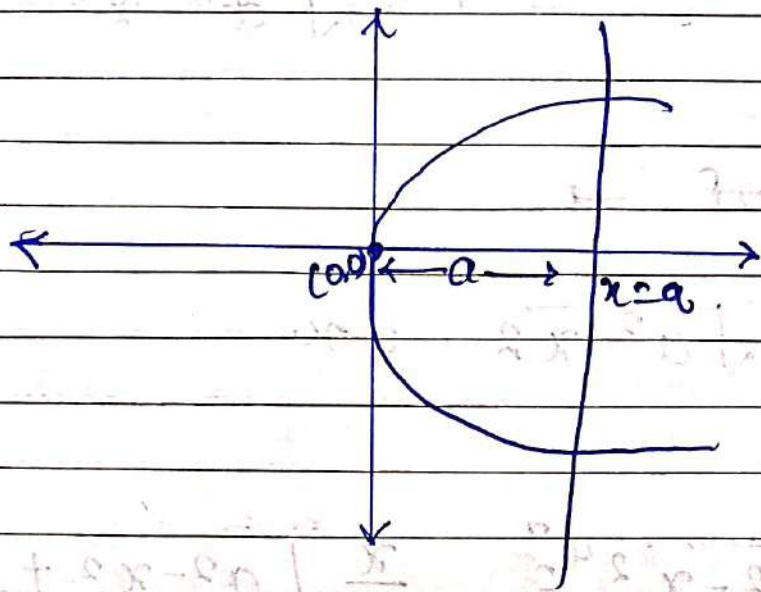
$$\Rightarrow 4 \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$\Rightarrow \pi a^2$$

Ans.

Q:- Find the area bounded by the parabola $y^2 = 4ax$ and $x = a$

Soln:-



Total area bounded by the parabola

$$= 2 \int_0^a \sqrt{4ax} \, dx$$

$$= 2 \times \int_0^a 2\sqrt{a} \cdot \sqrt{x} \, dx$$

$$= 4\sqrt{a} \int_0^a \sqrt{x} \, dx$$

$$\Rightarrow 4\sqrt{a} \int_0^a (x)^{1/2} dx$$

$$\Rightarrow 4\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^a$$

$$\Rightarrow \frac{8\sqrt{a}}{3} \left[x^{3/2} \right]_0^a$$

$$\Rightarrow \frac{8\sqrt{a}}{3} \cdot a^{3/2}$$

$$\Rightarrow \frac{8}{3} \cdot (a)^{1/2} \cdot (a)^{3/2}$$

$$\Rightarrow \frac{8}{3} \sqrt{a}^2 \text{ Sq unit.}$$

Ans

Q.10 - Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and $y + z = 4$ and $z = 0$

$$x^2 + y^2 = 4$$

$$y + z = 4$$

$$z = 0$$

for z limit

$$z = 0 \text{ to } z = 4 - y$$

for y limit

$$y^2 = 4 - x^2$$

$$y = \pm \sqrt{4 - x^2}$$

$$y = -\sqrt{4 - x^2} \text{ to } y = \sqrt{4 - x^2}$$

for limit of x

$$x^2 + y^2 = 4$$

$$\therefore y = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$$x = -2 \text{ to } x = 2$$

Volume

$$\rightarrow \int \int \int dv$$

$$\Rightarrow \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dx dy dz$$

$$\Rightarrow \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{4-y} dz$$

$$\Rightarrow \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dz$$

$$\Rightarrow \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy [z]_0^{4-y}$$

$$\Rightarrow \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y) \cdot dy$$

$$\Rightarrow \int_{-2}^2 dx \left[4-y \right] \left[4y - \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$\int_{-2}^2 dx \left[4(\sqrt{4-x^2}) - \frac{(\sqrt{4-x^2})^2}{2} \right] - \left[-4\sqrt{4-x^2} \right.$$

$$\left. - \frac{(-\sqrt{4-x^2})^2}{2} \right]$$

$$\int_{-2}^2 dx \left[4\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^2}{2} + 4\sqrt{4-x^2} + \frac{(\sqrt{4-x^2})^2}{2} \right]$$

$$\int_{-2}^2 dx \quad 8\sqrt{4-x^2}$$

$$8 \int_{-2}^2 \sqrt{4-x^2} \cdot dx$$

$$8 \int_{-2}^2 \sqrt{(2)^2 - x^2} \cdot dx$$

$$\left[\int \sqrt{a^2 - x^2} = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C \right]$$

$$\Rightarrow 8 \left[\frac{x}{2} \sqrt{4 - x^2} + 2 \sin^{-1}\left(\frac{x}{2}\right) \right]_2^{-2}$$

$$\Rightarrow 8 \left[\left\{ \frac{2}{2} \sqrt{4-4} + 2 \sin^{-1}\left(\frac{2}{2}\right) \right\} - \left\{ \frac{-2}{2} \sqrt{4-4} + 2 \sin^{-1}\left(\frac{-2}{2}\right) \right\} \right]$$

$$\Rightarrow 8 \left[\left\{ 2 \times \frac{\pi}{2} \right\} - \left\{ -2 \times \frac{\pi}{2} \right\} \right]$$

$$\Rightarrow 8 \{ \pi + \pi \} = 16\pi$$

Ans.

UNIT - III

FOURIER SERIES

* Fourier Series

Let function $f(x)$ is defined as periodic in the interval (a, b) then its fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) \cdot dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cdot \cos\left(\frac{2n\pi x}{b-a}\right) \cdot dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \cdot \sin\left(\frac{2n\pi x}{b-a}\right) \cdot dx$$

Q. Find the fourier series representing $f(x) = x^2$ for interval $-\pi < x < \pi$

Soln:-

$$f(x) = x^2$$

$$a = -\pi, \quad b = \pi$$

$$b-a = \pi + \pi = 2\pi$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) \cdot dx$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} x^2 \cdot dx$$

$$\left[\therefore \int_{-a}^a f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx \right]$$

$$\Rightarrow a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \cdot dx$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0 \right]$$

$$\Rightarrow a_0 = \frac{2}{\pi} \times \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cdot \cos \left(\frac{2n\pi x}{b-a} \right) \cdot dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cdot \cos \left(\frac{2n\pi x}{2\pi} \right) \cdot dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cdot \cos nx \cdot dx$$

$$\Rightarrow \frac{2}{\pi} \left[x^2 \cdot \frac{\sin nx}{n} - 2x \left(\frac{\cos nx}{n \times n} \right) + 2 \left(\frac{\sin nx}{n \times n \times n} \right) \right]$$

$$\Rightarrow \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2 \sin n\pi}{n^3} \right]$$

$$\left[\because \sin n\pi = 0 \quad \star \cos n\pi = (-1)^n \right]$$

$$\Rightarrow \frac{2}{\pi} \left[\frac{\pi^2 \cdot 0}{n} + \frac{2\pi (-1)^n}{n^2} - 2 \times \frac{0}{n^3} \right]$$

$$\Rightarrow \frac{2}{\pi} \left[\frac{2\pi (-1)^n}{n^2} \right]$$

$$\Rightarrow \frac{2}{\pi} \times \frac{2\pi (-1)^n}{n^2} \Rightarrow$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \cdot \sin \left(\frac{2n\pi x}{b-a} \right) \cdot dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} x^2 \cdot \sin \left(\frac{2n\pi x}{2\pi} \right) \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cdot \sin nx \cdot dx$$

Even $\times$ odd = odd

$$= \frac{2}{\pi} \left[x^2 \cdot \frac{-\cos nx}{n} - 2x \cdot \frac{-\sin nx}{n \times n} + 2 \cdot \frac{\cos nx}{n \times n \times n} \right]_{\pi}^0$$

$$= \frac{2}{\pi} \left[-x^2 \frac{\cos nx}{n} + 2x \frac{\sin nx}{n^2} + 2 \frac{\cos nx}{n^3} \right]_0^{\pi}$$

$$b_n = 0$$

Fourier Series:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{b-a}\right) + b_n \sin\left(\frac{2\pi n x}{b-a}\right)$$

$$f(x) = \frac{2\pi^2}{3} \times \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cdot \cos nx + 0$$

$$f(x) = \frac{\pi^2}{3} + \frac{4(-1)^1}{1^2} \cos x + \frac{4(-1)^2}{2^2} \cos 2x + \dots$$

$$f(x) = \frac{\pi^2}{3} - 4 \cos x + \cos 2x + \dots$$

Ans.

Question:- Obtain the Fourier series for the function $f(x) = x$ in the interval $(-\pi, \pi)$.

Soln:-

$$f(x) = x$$

$$a = -\pi, b = \pi$$

$$b-a = \pi - (-\pi) = 2\pi$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x) \cdot dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} x \cdot dx$$

$$= \frac{2}{2\pi} \int_0^{\pi} x \cdot dx$$

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$$a_n = \frac{2}{b-a} \int_a^b f(x) \cdot \cos\left(\frac{2\pi nx}{b-a}\right) \cdot dx$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} x \cdot \cos\left(\frac{2\pi nx}{b-a}\right) \cdot dx$$

$$a_n = \frac{1}{\pi} \times 0 = 0$$

$$b_n = \frac{2}{b-a} \int_a^b x \cdot \sin\left(\frac{2\pi nx}{b-a}\right) \cdot dx$$

$$= \frac{2}{2\pi} \int_{-\pi}^{\pi} x \cdot \sin nx \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cdot \sin nx \cdot dx$$

$$= \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi \frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right]$$

$$= \frac{2}{\pi} \times -\pi \frac{(-1)^n}{n} = -2 \frac{(-1)^n}{n} = b_n$$

Obtain Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{b-a}\right) + b_n \sin\left(\frac{2\pi nx}{b-a}\right)$$

$$f(x) = 0 + \sum_{n=1}^{\infty} 0 + \frac{-2(-1)^n}{n} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx \dots$$

$$= \frac{(-2)(-1)^1 \sin x}{1} + \frac{(-2)(-1)^2 \sin 2x}{2} + \dots$$

$$f(x) = 2 \sin x - \sin 2x + \dots$$

Ans.

* Half Range Fourier Cosine Series

The half range Fourier cosine series in $(0, l)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l} \int_0^l f(x) \cdot dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) \cdot dx$$

* Half range of Fourier ^{Sine} Cosine Series

The half range Fourier sine series in $(0, l)$ is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

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Q:- Express $f(x) = x$ as a half range cosine series in $0 \leq x < 2$

Soln:- $f(x) = x$
Interval $- (0, 2) \rightarrow (0, 1)$

$$a_0 = \frac{2}{l} \int_0^l f(x) \cdot dx$$

$$= \frac{2}{2} \int_0^2 x \cdot dx$$

$$= \int_0^2 x \cdot dx$$

$$= \left[\frac{x^2}{2} \right]_0^2$$

$$a_0 = 2$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$a_n = \int_0^2 x \cdot \cos\left(\frac{n\pi x}{2}\right)$$

$$= \left[\frac{x \sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - \frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= \left[\frac{x \sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

$$= \left[\frac{2 \sin\left(\frac{2n\pi}{2}\right)}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{2n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right] - \left[0 + \frac{\cos 0}{\left(\frac{n\pi}{2}\right)^2} \right]$$

$$= \frac{2 \sin n\pi}{\frac{n\pi}{2}} + \frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2} - \frac{1}{\left(\frac{n\pi}{2}\right)^2}$$

$$= \cos \frac{(-1)^n}{\left(\frac{n\pi}{2}\right)^2} - \frac{1}{\left(\frac{n\pi}{2}\right)^2}$$

$$= \frac{(-1)^n - 1}{\left(\frac{n\pi}{2}\right)^2}$$

$$= \left(\frac{2}{n\pi}\right)^2 [(-1)^n - 1]$$

$$a_n = \frac{4}{n^2 \pi^2} [(-1)^n - 1]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \cdot dx$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [(-1)^n - 1] \cos\left(\frac{n\pi x}{2}\right)$$

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$$1 + \frac{4}{1^2 \pi^2} [(-1)^1 - 1] \cos \frac{\pi x}{2} + \frac{4}{2^2 \pi^2} [(1)^2 - 1] \cos \frac{2\pi x}{2} \\ + \frac{4}{3^2 \pi^2} [(-1)^3 - 1] \cos \frac{3\pi x}{2} \dots$$

Ans.

* Vector Space:

Algebraic structure

→ Ordered pair of set and binary operation

→ Ex $(\mathbb{R}, +)$

Group → closure, Associative, Identity, Inverse

Abelian group → closure, Associative, Identity, Inverse, Commutative

Q:- Prove that $(\mathbb{R}, +)$ is an abelian groupSoln:- $\mathbb{R} = \{-\infty, \dots, \infty\}$
 $+ \rightarrow$ Binary operation

① closure property

$$a = 1, b = 3$$

$$* \rightarrow +$$

$$\Rightarrow a * b$$

$$\Rightarrow a + b = 1 + 3 = 4$$

$$a, b \in \mathbb{R}$$

$$a * b \in \mathbb{R}$$

② Associative property

$$a * (b * c) = (a * b) * c$$

$$a = 1, b = 2, c = 3$$

$$* \rightarrow +$$

$$1 + (2 + 3) = (1 + 2) + 3$$

$$1 + 5 = 3 + 3$$

$$6 = 6$$

(ii) Identity Element

$$a * e = a$$

$$a = 4$$

$$4 * e = a$$

$$4 + e = 4$$

$$e = 0$$

→ Identity element

(iii) Inverse

$$a * b = e$$

$$4 + (-4) = 0$$

$$b = -4$$

(iv) Commutative

$$a * b = b * a$$

$$a = 2, b = 3$$

$$2 + 3 = 3 + 2$$

$$5 = 5$$

(v) Internal Composition

$$\text{Ex: } (B, +)$$

$$x_1 \text{ and } x_2 \in B$$

$$x_1 + x_2 \in B$$

→ Internal Composition

* External Composition:-

Algebraic Structure

$(A, +)$ and $(B, +)$

$\downarrow$
 α

$\downarrow$
 β

$\alpha \in A$ and $\beta \in B$

$\alpha \cdot \beta = \gamma \in A$

External composition of set A.

Ex:-

$(A, +)$, $(B, +, \cdot)$

$A = \{1, 2, 3, 4, \dots\}$, $B = \{1, 3, 5, \dots\}$

$\alpha = 2 \in A$, $\beta = 1 \in B$

$\alpha \cdot \beta = 2 \cdot 1 = 2 \in A$

* Vector space

$V \rightarrow$ set $\rightarrow$ Vectors

$F \rightarrow$ set $\rightarrow$ Scalars

$V(F) \rightarrow$ Vector space of field F

$(V, +)$ is an abelian group

$\rightarrow V(F)$ is external composition

Q:- Check whether the vector $V = \{x, y, z \in \mathbb{R}^3 : 2x + 3y^2 - 4z^2 = 0\}$ is vector space or not?

Soln

$$2x + 3y^2 - 4z^2$$

$$\text{put } (2, 0, 1)$$

$$\Rightarrow 2(2) + 3(0)^2 - 4(1)^2$$

$$\Rightarrow 4 - 4 = 0$$

$(2, 0, 1) \rightarrow$ Vector space - I

$$\Rightarrow 2x + 3y^2 - 4z^2$$

$$\text{put } (8, 0, 2)$$

$$\Rightarrow 2(8) + 0 - 4(2)^2$$

$$= 16 + 0 - 16 = 0$$

$(8, 0, 2) \rightarrow$ Vector space - II

adding V.S I + V.S II

$$\rightarrow (2, 0, 1) + (8, 0, 2)$$

$$\Rightarrow (10, 0, 3) \rightarrow \text{V.S - III}$$

Verify

$$2x + 3y^2 - 4z^2$$

$$\text{put } (10, 0, 3)$$

$$\Rightarrow 20 + 0 - 36$$

$$= -16 \neq 0$$

$\therefore$ Additional vector does not belong to V
 $\therefore V$ is not a vector space

Q1- check whether the vector $V =$

$\{a, b, c \in \mathbb{R}^3; (a-b)c = 0\}$ is vector space or not.

Soln: $\forall$ for vector space-I

$$(a-b)c = 0$$

$$\Rightarrow (a-b)c$$

$$\text{put } a=1, b=1, c=2$$

$$\Rightarrow (1-1)2 = 0$$

$$V.S - I :- (1, 1, 2)$$

for $V.S - II$

$$(a-b)c = 0$$

$$\Rightarrow (a-b)c$$

$$\text{put } a=4, b=9, c=0$$

$$\Rightarrow (4-9) \times 0 = 0$$

$$V.S - II :- (4, 9, 0)$$

Here we form two vector space which belong to V

$$V.S - I + V.S - II$$

$$(1, 1, 2) + (4, 9, 0)$$

$$\Rightarrow (1+4, 1+9, 2+0)$$

$$\Rightarrow (5, 10, 2)$$

Verify

$$\rightarrow (a-b)c$$

$$\rightarrow (5, 10, 2)$$

$$\rightarrow (5-10)2$$

$$= -5 \times 2 = -10 \neq 0$$

$\therefore V-S-\text{III}$ does not belong to V
 $\therefore V$ is not vector space

* Vector Subspace

$\rightarrow V(F)$ be a vector space

$\rightarrow W$ is the subset of V

$\rightarrow W$ itself a vector space

$\rightarrow W$ is called vector subspace of $V(F)$

* Theorem:- A subset W of a vector space $V(F)$ is a subspace if and only if
 $a, b \in F$ and $\alpha, \beta \in W$ then
 $a\alpha + b\beta \in W$
 OR

The necessary and sufficient condition for non empty subset of V is
 $\forall a, b \in F, \forall \alpha, \beta \in W$ then
 $a\alpha + b\beta \in W$

Proof

Necessary Condition

Let w be the subspace of $V(F)$

$$\alpha \in w \quad \text{and} \quad a \in F$$

$$a \cdot \alpha \in w \quad \text{--- (i)}$$

$$\beta \in w \quad \text{and} \quad b \in F$$

$$b \cdot \beta \in w \quad \text{--- (ii)}$$

w is the subspace of $V(F)$

$\therefore w$ is closed under vector addition

$$\boxed{a\alpha + b\beta \in w}$$

there

$$\alpha, \beta \in w \quad \text{and} \quad a, b \in F$$

Q:- Show that the set $w = \{(a, b, 0) ; a, b \in F\}$ is a subspace of $V_3 F$

Soln $w = \{(a, b, 0) : a, b \in F\}$

when $a, b \in F$ then $(a, b, 0) \in w$

$$\text{let } \alpha = (a_1, b_1, 0) \quad \text{and} \quad \beta = (a_2, b_2, 0)$$

from theorem

$$a\alpha + b\beta$$

$$\Rightarrow a(a_1, b_1, 0) + b(a_2, b_2, 0)$$

$$\Rightarrow (aa_1, ab_1, 0) + (ba_2, bb_2, 0)$$

$$\Rightarrow (aa_1 + ba_2, ab_1 + bb_2, 0)$$

$$\therefore aa_1 + ba_2 \in F$$

$$\therefore ab_1 + bb_2 \in F$$

$$\therefore (aa_1 + ba_2, ab_1 + bb_2, 0) \in w$$

$$\therefore a\alpha + b\beta \in w$$

Proved

$\therefore w$ is the subspace of $V_3(F)$.

* Theorem-2

The necessary and sufficient condition for non-empty subset w for a vector space $V(F)$ to be a subspace of V are

$$(i) \alpha \in w, \beta \in w, \alpha - \beta \in w, \alpha + \beta \in w$$

$$a \in F, \alpha \in w$$

$$\Rightarrow a\alpha \in w$$

* If V is a set of all $(n \times n)$ matrices over the field F then show that a set w of all $(n \times n)$ symmetric matrices form a vector subspace of $V(F)$.

Proof

Let $A = [a]_{n \times n}$ and $B = [b]_{n \times n}$ be the symmetric matrix then we have

$$A + B = [a]_{n \times n} + [b]_{n \times n}$$

$$= [a+b]_{n \times n}$$

Symmetric matrix
 $(a+b)' = [A+B]'$

$\therefore$ Sum of symmetric matrix is symmetric

$\therefore$ W is closed under Addition

$a \in F, A = [a]_{n \times n} \in W$

$$aA = a[a]_{n \times n}$$

$$= [a, a]_{n \times n}$$

$$= (aA)'$$

aA is symmetric matrix

W is closed under multiplication

$\therefore W(F)$ is vector subspace of $V(F)$

Proved

* Linear Combination

$$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in V(F)$$

$$a_1, a_2, a_3, a_4, \dots, a_n \in F$$

Linear Combination

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \dots + \alpha_n a_n = \alpha \in V(F)$$

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots = \sum_{i=1}^n \alpha_i a_i$$

* Linear span of a set

Linear span of a set $S \equiv L(S)$

$$L(S) = \left\{ \sum_{i=1}^n \alpha_i a_i : \begin{matrix} \alpha_i \in S \\ a_i \in F \end{matrix} \right\}$$

* Linear sum of two subspace

$$W_1 + W_2 = \{ \alpha + \beta : \begin{matrix} \alpha \in W_1 \\ \beta \in W_2 \end{matrix} \}$$

* Direct sum of two subspace

$$W_1 + W_2 = \{ \alpha_1 + \beta_1 : \begin{matrix} \alpha_1 \in W_1, \beta_1 \in W_2 \\ \alpha_1 + \beta_1 \text{ is unique} \end{matrix} \}$$

* Linear Independent :

$\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are said to be linearly independent if for scalars $a_1, a_2, a_3, \dots, a_n \in F$

$$a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + \dots + a_n \alpha_n = 0$$

$$\underline{a_1 = a_2 = a_3 = \dots = a_n = 0}$$

Linear
Independent

$$\underline{a_1 = a_2 = a_3 \neq 0}$$

Linear
dependent

Q:- Check the vectors $\alpha_1 = (1, 2, 3)$, $\alpha_2 = (1, 0, 0)$, $\alpha_3 = (0, 1, 0)$ and $\alpha_4 = (0, 0, 1)$ in $V_3(\mathbb{R})$ are linearly dependent or independent.

Soln:- $\alpha_1 = (1, 2, 3)$, $\alpha_2 = (1, 0, 0)$, $\alpha_3 = (0, 1, 0)$, $\alpha_4 = (0, 0, 1)$

let a_1, a_2, a_3 and a_4 are scalars $\in F$

$$a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + a_4 \alpha_4 = 0$$
$$\Rightarrow a_1(1, 2, 3) + a_2(1, 0, 0) + a_3(0, 1, 0) + a_4(0, 0, 1) = 0$$

$$a_1 + a_2 = 0 \quad \text{--- (i)}$$

$$2a_1 + a_3 = 0 \quad \text{--- (ii)}$$

$$3a_1 + a_4 = 0 \quad \text{--- (iii)}$$

put $a_1 = 1$ in eq (i) and (ii)

$$0 + a_2 = 0$$

$$\therefore a_2 = 0$$

$$2a_1 + a_3 = 0$$

$$2 + a_3 = 0$$

$$\therefore a_3 = -2$$

$$3(1) + a_4 = 0$$

$$\therefore a_4 = -3$$

$$a_1 = 1, a_2 = 0, a_3 = -2, a_4 = -3$$

$$a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, a_4 \neq 0$$

$\therefore$ this is linearly dependent vector

Ans



Matrix - A

Rank of A

Rank of A =
no. of scalar

Rank of A < No of
Scalars

Vector = Linearly
Independent

Vector = linearly
dependent

Q:- Find whether the vectors $(2, 3, -1)$, $(-1, 4, -2)$ and $(1, 18, -4)$ are linearly independent or dependent.

Soln:- $\alpha_1 = (2, 3, -1)$, $\alpha_2 = (-1, 4, -2)$, $\alpha_3 = (1, 18, -4)$

Let scalars be
 a_1, a_2, a_3
Linear Combination

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$a_1(2, 3, -1) + a_2(-1, 4, -2) + a_3(1, 18, -4) = 0$$

$$(2a_1, 3a_1, -a_1) + (-a_2, 4a_2, -2a_2) + (a_3, a_3 \cdot 18, -4a_3) = 0$$

$$2a_1 - a_2 + a_3 = 0 \quad \text{--- (i)}$$

$$3a_1 + 4a_2 + 18a_3 = 0 \quad \text{--- (ii)}$$

$$-a_1 - 2a_2 - 4a_3 = 0 \quad \text{--- (iii)}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 18 \\ -1 & -2 & -4 \end{bmatrix}$$

$$C_1 \leftrightarrow C_2$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 4 & 3 & 18 \\ -2 & -1 & -4 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 4R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 11 & 22 \\ 0 & -5 & -6 \end{bmatrix}$$

$$R_3 \cdot C_2 \rightarrow 6C_2 - 5C_3$$

$$\begin{bmatrix} -1 & 7 & 1 \\ 0 & -44 & 22 \\ 0 & 0 & -6 \end{bmatrix} \begin{array}{l} \rightarrow \text{Non Zero} \\ \rightarrow \text{Non Zero} \\ \rightarrow \text{Non Zero} \end{array}$$

$$\text{Rank of } A = 3$$

$\therefore$ Rank of $A =$ No of Scalars
Vectors are linearly independent

Q:- Find wheather the set of vectors
 $V_1 = (1, 2, 1)$, $V_2 = (3, 1, 5)$, $V_3 = (3, 4, 7)$
is linearly independent or dependent.

Soln:- Given

$$V_1 = (1, 2, 1), V_2 = (3, 1, 5), V_3 = (3, 4, 7)$$

let scalars be

$$a_1, a_2, a_3$$

Linear Combination

$$V_1 a_1 + V_2 a_2 + V_3 a_3 = 0$$

$$(1, 2, 1) a_1 + (3, 1, 5) a_2 + (3, 4, 7) a_3 = 0$$

$$(a_1, 2a_1, a_1) + (3a_2, a_2, 5a_2) + (3a_3, 4a_3, 7a_3) = 0$$

$$a_1 + 3a_2 + 3a_3 = 0 \quad \text{--- (i)}$$

$$2a_1 + a_2 - 4a_3 = 0 \quad \text{--- (ii)}$$

$$a_1 + 5a_2 + 7a_3 = 0 \quad \text{--- (iii)}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 2 & 4 \end{bmatrix}$$

$$C_2 \rightarrow 2C_2 - C_3$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & -10 \\ 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \rightarrow 5R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{New Zero} \\ \rightarrow \text{Non Zero} \\ \rightarrow \text{Zero} \end{array}$$

$$\text{Rank of } A = 2$$

$$\text{No of Scalar} = 3$$

$$\text{Rank of } A \neq \text{No. of Scalar}$$

$\therefore$ linearly dependent

Q:- Are the vectors $(2, 2, 2, 4)$, $(2, -2, -4, 0)$, $(4, -2, -5, 2)$, $(4, 2, 1, 6)$ linearly independent.

Ans:- $\alpha_1 = (2, 2, 2, 4)$, $\alpha_2 = (2, -2, -4, 0)$

$\alpha_3 = (4, -2, -5, 2)$, $\alpha_4 = (4, 2, 1, 6)$

Let scalar's be

a_1, a_2, a_3, a_4

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 = 0$$

$$(2, 2, 2, 4)a_1 + (2, -2, -4, 0)a_2 + (4, -2, -5, 2)a_3 + (4, 2, 1, 6)a_4 = 0$$

$$(2a_1, 2a_1, 2a_1, 4a_1) + (2a_2, -2a_2, -4a_2, 0) + (4a_3, -2a_3, -5a_3, 2a_3) + (4a_4, 2a_4, a_4, 6a_4) = 0$$

$$2a_1 + 2a_2 + 4a_3 + 4a_4 = 0 \quad \text{--- (i)}$$

$$2a_1 - 2a_2 - 2a_3 + 2a_4 = 0 \quad \text{--- (ii)}$$

$$2a_1 - 4a_2 - 5a_3 + a_4 = 0 \quad \text{--- (iii)}$$

$$4a_1 + 0a_2 + 2a_3 + 6a_4 = 0 \quad \text{--- (iv)}$$

$$\begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & -2 & -2 & 2 \\ 2 & -4 & -5 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix}$$

$$C_1 \rightarrow C_1/2$$

$$C_2 \rightarrow C_2/2$$

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & 2 \\ 1 & -2 & -5 & 1 \\ 2 & 0 & 2 & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & -3 & -9 & -3 \\ 0 & -2 & -6 & -2 \end{bmatrix}$$

$$R_2 \rightarrow -R_2$$

$$R_3 \rightarrow -R_3$$

$$R_4 \rightarrow -R_4$$

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 3 & 9 & 3 \\ 0 & 2 & 6 & 2 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 - 3R_2$$

$$R_4 \rightarrow R_4 - R_2$$

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R_4 \rightarrow 3R_4 - R_3$$

$$\begin{bmatrix} 1 & 1 & 4 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Rank of } A = 3$$

$$\text{Rank of } A \stackrel{\text{no.}}{<} \text{Scalar of}$$

$\therefore$ this is not linearly independent
Ans

Qy. Discuss the maximum and minimum of $x^3 + y^3 - 3xy$.

Sol $\rightarrow$ let $u = x^3 + y^3 - 3xy$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y \quad \text{and} \quad \frac{\partial u}{\partial y} = 3y^2 - 3x = 0$$

for maxima or minima of u , putting

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$$3x^2 - 3y = 0 \quad \text{and} \quad 3y^2 - 3x = 0$$

$$\text{We get } x^2 = y \quad \text{and} \quad y^2 = x \Rightarrow x = y = 1$$

The point $(1,1)$ is a critical point.

$$r = \frac{\partial^2 u}{\partial x^2} = 6x, \quad S = \frac{\partial^2 u}{\partial x \partial y} = -3, \quad t = \frac{\partial^2 u}{\partial y^2} = 6y$$

at point $(1,1)$ we have

$$r = 6, \quad S = -3, \quad t = 6$$

$$\therefore rt - S^2 = 36 - 9 = 27 > 0$$

$$r > 0$$

Hence u has minimum at $x=y=1$

$$\text{Min Value of } u \text{ i.e. } u_{\min} = [x^3 + y^3 - 3xy]_{x=y=1}$$

$$u_{\min} = 1 + 1 - 3 = -1$$

Q4. Test for maxima or minima given that $f(x,y) = x^2 + 2xy + 2y^2 + 2x + y$

Sol: $\rightarrow F = x^2 + 2xy + 2y^2 + 2x + y$

$$\frac{\partial f}{\partial x} = 2x + 2y + 2, \quad \frac{\partial f}{\partial y} = 2x + 4y + 1$$

$$r = 2, \quad S = 2, \quad t = 4$$

for max or min we have

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$2x + 2y + 2 = 0 \text{ and } 2x + 4y + 1 = 0$$

$$x = -3/2, \quad y = 1/2$$

at point $(-\frac{3}{2}, \frac{1}{2})$ we have $r - s^2$
 $2 \times 4 - (2)^2$
 $8 - 4 = 4 > 0$
 $r > 0$

$f(x, y)$ has min value at $x = -\frac{3}{2}, y = \frac{1}{2}$

$$F_{\min} = \frac{9}{4} + 2\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)$$

$$F_{\min} = \frac{-5}{4} \quad \text{Ans.}$$

Qy. find the maximum or minimum value of the function $x^3 y^2 (1-x-y)$.

Sol: $\rightarrow u = x^3 y^2 (1-x-y)$

$$\frac{\partial u}{\partial x} = 3x^2 y^2 (1-x-y) - x^3 y^2 = 0 \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = 2x^3 y (1-x-y) - x^3 y^2 = 0 \rightarrow (2)$$

Subtracting eq (2) from (1), we get

$$x^2 y (1-x-y) (3y - 2x) = 0$$

$$\Rightarrow y = \frac{2x}{3}$$

Putting the value of y in eq (1) we get

$$3x^2 \cdot \frac{4}{9} x^2 \left(1-x-\frac{2x}{3}\right) - x^3 \cdot \frac{4}{9} x^2 = 0$$

$$\frac{4}{9} x^4 [3 - 5x - x] = 0$$

$$\frac{4}{9} x^4 [3 - 6x] = 0$$

$$x = \frac{1}{2}$$

and putting the value of x in $y = \frac{2}{3}x$

$$\text{we get } \Rightarrow y = \frac{1}{3}$$

$$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y - 6xy^3$$

$$\text{the value of } r \text{ at point } \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{9}$$

$$t = 2x^3 - 2x^4 - 6x^2y$$

$$\text{the value of } t \text{ at point } \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{8}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -6x^2y - 8x^3 - 9x^2y^2$$

$$\text{the value of } s \text{ at point } \left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{12}$$

$$rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = +ve$$

hence u has a maximum value

$$\text{at } x = \frac{1}{2}, y = \frac{1}{3}$$

$$\text{Maxi value of } u = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

Eigen Vectors : $\rightarrow$ Let A be any square matrix. If there exists a non zero column vector

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a eigen value λ such that

$$[A - \lambda I]X = 0$$

$$\Rightarrow AX = \lambda X$$

then X is called an eigen vector or characteristic vector of A corresponding to the eigen value λ .

Q Determine Eigen Vector of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol : $\rightarrow \lambda = 1, 6$

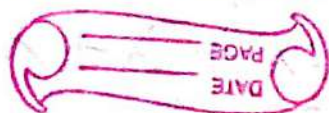
Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector corresponding to eigen values λ is $[A - \lambda I]X = 0$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

When $\lambda = 1$

then eq. (1) becomes

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Since rank of coefficient matrix is 1, So taking

any one equation, Say

$$4x + 4y = 0$$

$$x + y = 0$$

$$\Rightarrow x = -y \Rightarrow \frac{x}{1} = \frac{y}{-1} = K$$

$$X_1 = \begin{bmatrix} K \\ -K \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ is eigen vector for } \lambda = 1$$

When $\lambda = 6$ then eq (1) becomes

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 4y = 0$$

$$x = 4y$$

$$\frac{x}{4} = \frac{y}{1} = K$$

$$X_2 = \begin{bmatrix} 4K \\ K \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ is eigen vector } \lambda = 6$$

Qy Find Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol: $\rightarrow \lambda = 0, 0, 3$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vectors for λ is given by $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

When $\lambda = 0$ (repeated roots) then eq (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly rank of coefficient matrix is 1,
So that selecting first row

$$x + y + z = 0 \text{ with } (n-r) = 3-1 = 2$$

2 variables are linearly independent

Suppose $y = k_1$, and $z = k_2$ then

$$x = -k_1 - k_2$$

$$X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

is eigen vector for $\lambda = 0$

When $\lambda = 3$ then eq (1) becomes

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since rank of Coefficient matrix is 2,
So that selecting first two rows

$$-2x + y + z = 0$$

$$x - 2y + z = 0$$

$$x = -y = z$$

$$\frac{1}{1+2} = \frac{-2-1}{-2-1} = \frac{4-1}{4-1}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

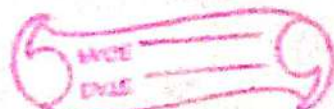
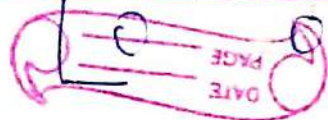
$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = K \text{ (Say)}$$

$$X = \begin{bmatrix} K \\ K \\ K \end{bmatrix} \text{ is a eigen vector for } \lambda = 3.$$

Q4 find the Eigen Values and Eigen Vectors.
of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Sol: $\rightarrow |A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$



$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda = 2, 3, 5$$

Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vectors corresponding to eigen values λ is given by $[A - \lambda I]x = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \textcircled{1}$$

When $\lambda = 2$ then eq(1) becomes

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since rank of coefficient matrix is 2,
So selecting any or first two rows
 $x + y + 4z = 0$

$$6z = 0$$

$$\frac{x}{6-0} = \frac{-y}{6-0} = \frac{z}{0-0}$$

$$\frac{x}{1} = \frac{-y}{-1} = \frac{z}{0} = K$$

$$x_1 = \begin{bmatrix} K \\ -K \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ is eigen vector for } \lambda = 2$$

When $\lambda = 3$ then eq (1) becomes

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since rank of coefficient matrix is 2,
so that selecting any or first two rows

$$0 \cdot x + y + 4z = 0$$

$$0 \cdot x - y + 6z = 0$$

$$\frac{x}{6+4} = \frac{-y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = K$$

$$x = K, \quad y = 0, \quad z = 0$$

$$X_2 = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is eigen vector for } \lambda = 3$$

When $\lambda = 5$ the eq (1) becomes

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since rank of coefficient matrix is 2,

so that selecting first two rows, we get

$$\frac{x}{6+12} = \frac{-y}{-12-0} = \frac{z}{6-0}$$

$$\frac{x}{18} = \frac{y}{12} = \frac{z}{6}$$



$$\frac{x}{3} = \frac{y}{2} = \frac{z}{1} = k$$

$$X_3 = \begin{bmatrix} 3K \\ 2K \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ is eigen vector for } \lambda = 5$$