CHAPTER 1: FUNDAMENTALS OF ENGINEERING MECHANICS

LEARNING OUTCOMES:

On completion of the subject, the student will be able to:

- Define and classify Mechanics
- Define and classify the forces and its system.
- Compute the force and apply it for solving problems on coplanar forces.
- Understand and apply resolution of forces.
- Understand composition of forces and apply it to solve problems
- Understand Moment of force, Varignon‟s theorem with applications,couple.

1.1 FUNDAMENTALS

ENGINEERING MECHANICS

Mechanics is that branch of physical science which deals with the action of forces on material bodies. Engineering Mechanics, which is very often referred to as Applied Mechanics, deals with the practical applications of mechanics in the field of engineering. Applications of Engineering Mechanics are found in analysis of forces in the components of roof truss, bridge truss, machine parts, parts of heat engines, rocket engineering, aircraft design etc.

DIVISIONS OF ENGINEERING MECHANICS

The subject of Engineering Mechanics may be divided into the following two main groups:

1. Statics and 2.Dynamics.

STATICS

It is the branch of Engineering Mechanics, which deals with the forces and their effects, while acting upon the bodies at rest.

DYNAMICS

It is the branch of Engineering Mechanics, which deals with the forces and their effects, while acting upon the bodies in motion. Dynamics may be further sub-divided into the following two branches:

1. Kinematics
2. Kinetics

Kinetic deals with the forces acting on moving bodies, whereas kinematics deals with the motion of the bodies without any reference to forces responsible for the motion.

FUNDAMENTAL UNITS

Every quantity is measured in terms of some internationally accepted units, called fundamental units.

All the physical quantities in Engineering Mechanics are expressed in terms of three fundamental quantities, *i.e*.

1. Length 2. Mass and 3. Time

DERIVED UNITS

Sometimes, the units are also expressed in other units (which are derived from fundamentalunits) known as derived units e.g. units of area, velocity, acceleration, pressure etc.

SYSTEMS OF UNITS

There are only four systems of units, which are commonly used and universally recognized. These are known as:

1. C.G.S. units**2.** F.P.S. units**3.** M.K.S. units and **4.** S.I. units.

In this study material we shall use only the S.I. system of units.

FUNDAMENTAL S.I UNITS

QUANTITIES	FUNDAMENTAL UNIT	SYMBOL
Length	Meter	m
Mass	Kilogram	Kg
Time	Second	S
Electric current	Ampere	A
Luminous intensity	Candela	Cd
Thermodynamic temperature	Kelvin	K

SOME S.I DERIVED UNITS

QUANTITIES	DERIVED UNIT	SYMBOL
Force	Newton	N
Moment	Newton-meter	Nm
Work done	Joule	J
Power	Watt	W
Velocity	Meter per second	m/s
Pressure	Pascal or Newton per square meter	Pa or N/m^2

MASS AND WEIGHT

Mass of a body is the total quantity of matter contained in the body.

Weight of a body is the force with which the body is attracted towards the centre of the earth.

DIFFERENCE BETWEEN MASS AND WEIGHT

MASS	WEIGHT
1. Mass is the total quantity of matter contained in a body.	1. Weight of a body is the force with which the body is attracted towards the centre of the earth.
2. Mass is a scalar quantity, because it has only magnitude and no direction.	2. Weight is a vector quantity, because it has both magnitude and direction.
3. Mass of a body remains the same at all places. Mass of a body will be the same whether the body is taken to the centre of the earth or to the moon.	3. Weight of body varies from place to place due to variation of „g‟ (i.e., acceleration due to gravity.
4. Mass resists motion in a body.	4. Weight produces motion in a body.
5. Mass can be measured by an ordinary balance.	5. It can be measured by a spring balance.

6. Mass of a body can never be zero.	6. Weight of a body can be zero.

RIGID BODY AND ELASTIC BODY

A body is said to be rigid if it does not undergo deformation whatever force may be applied to the body. In actual practice, there is no body which can be said to be rigid in true sense of terms.
A body is said to be elastic if it undergoes deformation under the action of force. All bodies are more or less elastic.

SCALAR AND VECTOR

All physical quantities can be divided into scalar quantity and vector quantity. Scalar quantity is that physical quantity which has only magnitude and no direction. For example, length, mass, energy etc.Vector quantity is that physical quantity which has both magnitude and direction. For example, force, velocity etc.

1.2 FORCE

FORCE SYSTEM

Force is that which changes or tends to change the state of rest of uniform motion of a body along a straight line. It may also deform a body changing its dimensions.The force may be broadly defined as an agent which produces or tends to produce, destroys or tends to destroy motion.Ithas a magnitude and direction.
Mathematically:

$$\text{Force} = \text{Mass} \times \text{Acceleration}.$$

Where F=force, M=mass and A=acceleration.

UNITS OF FORCE

In C.GS. System: In this system, there are two units of force: (1) Dyne and (ii) Gram force (gmf). Dyne is the absolute unit of force in the C.G.S. system. One dyne is that force which acting on a mass of one gram produces in it anacceleration ofone centimeter per second^2.
In M.K.S. System: In this system, unit of force is kilogram force (kgf). One kilogram force is that force which acting on a mass of one kilogram produces in it an acceleration of 9.81 m/ sec^2.
In S.I. Unit: In this system, unit of force is Newton (N). One Newton is that force which acting on a mass of one kilogram produces in it an acceleration of one m /sec^2.

1 Newton = 10^5 Dyne.

EFFECT OF FORCE

A force may produce the following effects in a body, on which it acts:

1. It may change the motion of a body. *i.e*. if a body is at rest, the force may set it in motion.And if the body is already in motion, the force may accelerate or decelerate it.
2. It may retard the forces, already acting on a body, thus bringing it to rest or in equilibrium.
3. It may give rise to the internal stresses in the body, on which it acts.
4. A force can change the direction of a moving object.
5. A force can change the shape and size of an object

CHARACTERISTICS OF A FORCE

In order to determine the effects of a force, acting on a body, we must know the followingcharacteristics of a force:

1. Magnitude of the force (*i.e*., 50 N, 30 N, 20N etc.)
2. The direction of the line, along which the force acts (*i.e.*, along *West,* at 30° North of East etc.). It is also known as line of action of the force.
3. Nature of the force (push or pull).
4. The point at which (or through which) the force acts on the body.

PRINCIPLE OF PHYSICAL INDEPENDENCE OF FORCES

It states, "If a number of forces are simultaneously acting on a particle, then the resultant of these forces will have the same effect as produced by all the forces"".

SYSTEM OF FORCES

When two or more forces act on a body, they are called to form a system of forces.Force system is basically classified into following types.

i. Coplanar forces

ii. Collinear forces

iii. Concurrent forces

iv. Coplanar concurrent forces

v. Coplanar non- concurrent forces

vi. Non-coplanar concurrent forces

vii. Non- coplanar non- concurrent force

COPLANAR FORCES: Theforces, whose lines of action lie on the same plane, are known as coplanar forces.

COLLINEAR FORCES: The forces, whose lines of action lie on the same line, are known ascollinear forces. They act along the same line. Collinear forces may act in the opposite directions or in the same direction.

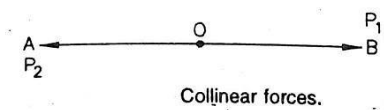
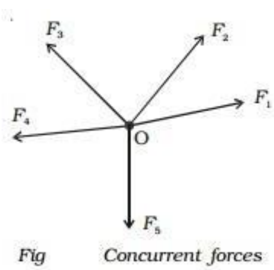


Fig 1.1

CONCURRENT FORCES: The forces, whose lines of action pass through a common point, are known as concurrent forces.The concurrent forces may or may not be collinear



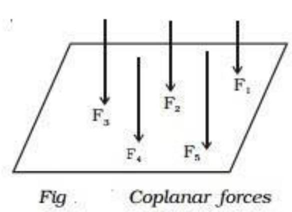
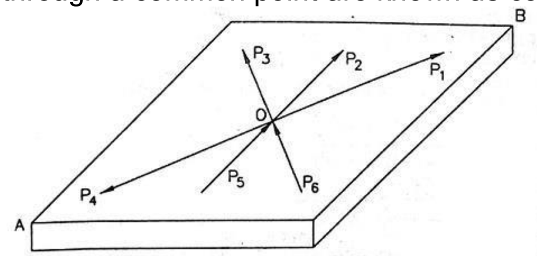


Fig. 1.2 Fig. 1.3

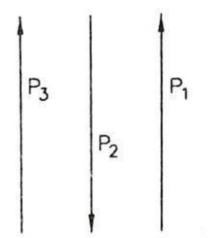
COPLANAR CONCURRENT FORCES: The forces, whose lines of action lie in the same plane and at the same time pass through a common point are known as coplanar concurrent forces.



Coplanar concurrent forces.

Fig 1.4

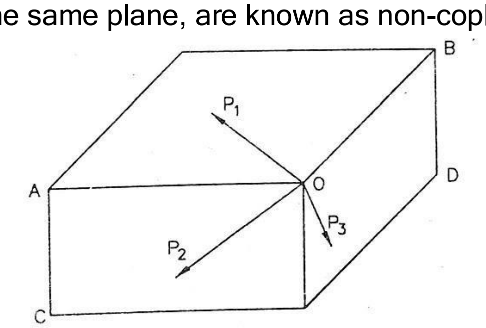
COPLANAR NON-CONCURRENT FORCES: The forces, which do not meet at one point, but their lines of action lie on the same plane, are known as coplanar non-concurrent forces.



Coplanar non-concurrent forces.

Fig 1.5

NON-COPLANAR CONCURRENT FORCES: The forces, which meet at one point, but their lines of action do not lie on the same plane, are known as non-coplanar concurrent forces.



Non-coplanar concurrent forces.

Fig 1.6

NON-COPLANAR NON-CONCURRENT FORCES: The forces, which do not meet at one point and their lines of action do not lie on the same plane, are called non-coplanar non-concurrent forces.

PULL AND PUSH: Pull is the force applied to a body at its front end to move the body in the direction of the force applied.
Push is the force applied to a body at its back end in order to move the body in the direction of the force applied.

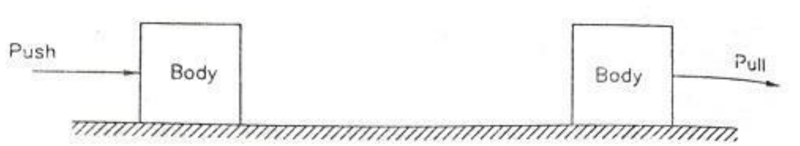


Fig 1.7 push and pull

ACTION AND REACTION: Action means active force. Reaction means reactive force. When a body having a weight W (=mg) is placed on a horizontal plane as shown in Fig 1.8, the body exerts a vertically downward force equal to „W‟ or „mg‟ on the plane. Then „W‟ is called action of the body on the plane. According to Newton‟s 3rd law of motion, every action has an equal and opposite reaction. But action and reaction never act on the same body. So, the horizontal plane will exert equal amount of force „R‟ on the body in the vertically upward direction. This vertically upward force acting on the body is called reaction of the plane on the body.

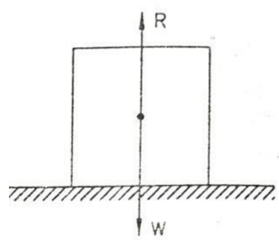
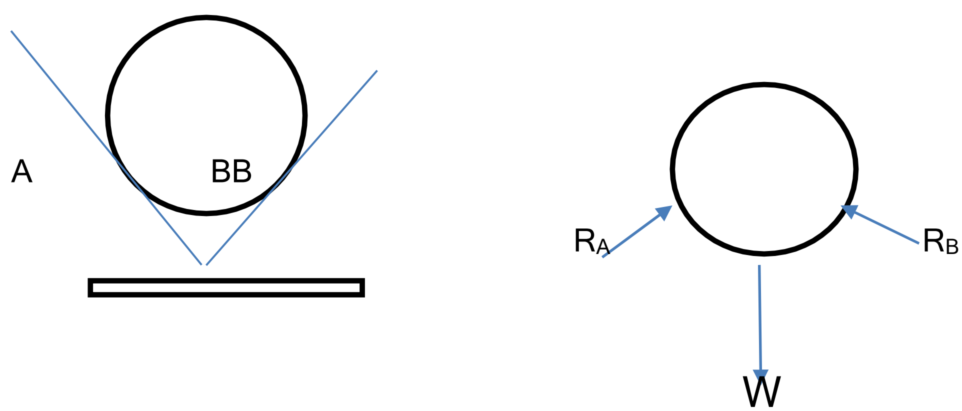


Fig 1.8 Action and reaction

FREE BODY DIAGRAM:

The representation of reaction force on the body by removing all the support or forces act from the body is called free body diagram.



Object with support Free Body Diagram

Fig.1.9

EXTERNAL FORCE AND INTERNAL FORCE: When a force is applied externally to a body; that force is called external force.

Internal force is that force which is set up in a body to resist deformation of the body caused by the external force.

TENSION: Tension is the pull to which a rope or wire or rod is subjected. In figure 1.10 (b) P is the tension applied to a rope.

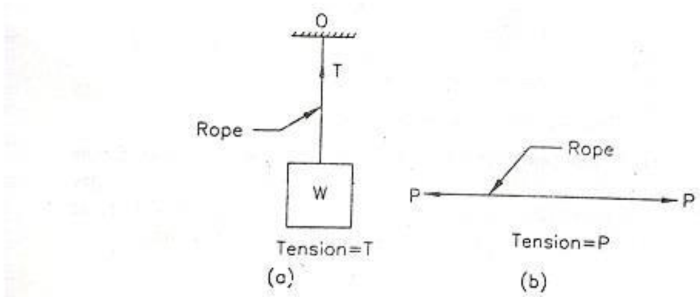


Fig 1.10 Tension

Let a body having weight W be suspended by means of a vertical rope fixed at its upper end at O. The point O is pulled downward by a force W. Hence the point O will exert equal amount of force W to the body, in the upward direction. This upward force on the rope is the tension of the rope. In Fig 1.10(a), T is the tension of the rope.

REPRESENTATION OF A FORCE

Since force is a vector quantity, it can be represented by a straight line. The length of the line represents magnitude of the force, the line itself represents the direction and an arrow put on the head of the straight line indicates the sense in which the force acts.

DENOTING A FORCE BY BOW'S NOTATION

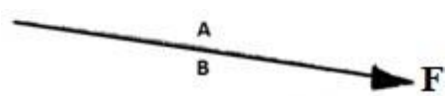


Fig 1.11

In Bow"s notation for denoting a force, two English capital letters are placed, one on each side of the line of action of the force. In figure 1.11 AB denotes the force F.

PRINCIPLE OF TRANSMISSIBILITY OF FORCES

It states, "If a force acts at any point on a rigid body, it may also be considered to act at anyother point on its line of action, provided this point is rigidly connected with the body."That means the point of application of a force can be moved anywhere along its line of action without changing the external reaction forces on a rigid body.

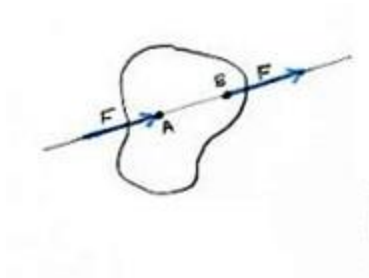


Fig 1.12

Here force at point A = force at B (the magnitude of force in the body at any point along the line of action are same)

PRINCIPLE OF SUPERPOSITION OF FORCES: This principle states that the combined effect of force system acting on a particle or a rigid body is the sum of effects of individual forces.

Consider two forces P and Q acting at A on a boat as shown in Fig 1.13. Let R be the resultant of these two forces P and Q. According to Newton‟s second law of motion, the boat will move in the direction of resultant force R with acceleration proportional to R. The same motion can be obtained when P and Q are applied simultaneously.

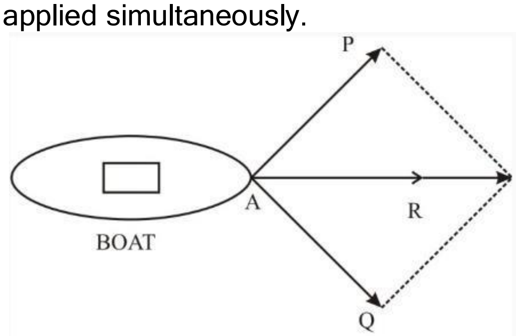
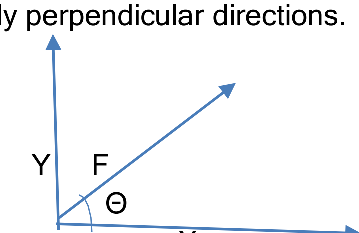


Fig 1.13

1.3 RESOLUTION OF A FORCE

RESOLUTION OF A FORCE

The process of splitting up the given force into a number of components, without changing its effect on the body is called resolution of a force. A force is, generally, resolved along two mutually perpendicular directions.



Fig(a)

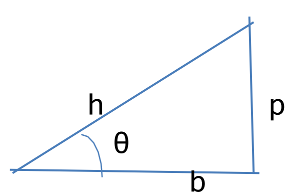


Fig (b)

Fig 1.14

(From Pythagoras theorem we know that

$\text{Sin}\theta = \frac{p}{h}$ => p= hsinθ similarly $\text{Cos}\theta = \frac{b}{h}$ => b= hcosθ)

By resolution of force F, we found

X=FCosθ and Y=F Sinθ

RESOLUTION OF A GIVEN FORCE INTO TWO COMPONENTS IN TWO ASSIGNED DIRECTION

Let P be the given force represented in magnitude and direction by OB as shown in Fig 1.15. Also let OX and OY be two given direction along which the components of P are to be found out.

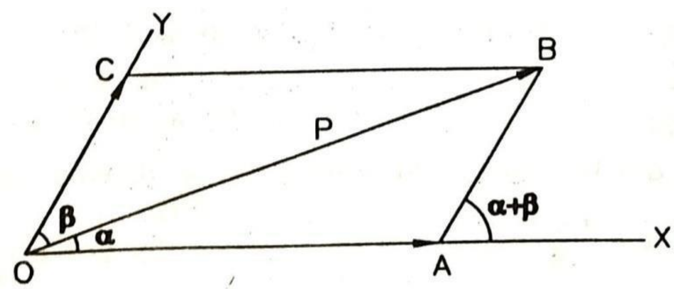


Fig 1.15

Let < BOX = α and < BOY = β

From B, lines BA and BC are drawn parallel to OY and OX respectively. Then the required components of the given force P along OX and OY are represented in magnitude and direction by OA and OC respectively. Since AB is parallel to OC, <BAX = <AOC = α + β

< AOB = 180º - (α + β)

Now, in Δ OAB

$$\frac{\text{OA}}{\text{Sin OBA}} = \frac{\text{AB}}{\text{Sin AOB}} = \frac{\text{OB}}{\text{Sin OAB}}$$

Or $\frac{\text{OA}}{\text{Sin}\,\beta} = \frac{\text{AB}}{\text{Sin}\,\alpha} = \frac{\text{OB}}{\text{Sin } 180° - (\alpha + \beta)}$

$$\frac{\text{OA}}{\text{Sin}\,\beta} = \frac{\text{AB}}{\text{Sin}\,\alpha} = \frac{\text{P}}{\text{Sin}\,(\alpha + \beta)}$$

$$\text{OA} = \frac{\text{P Sin}\,\beta}{\text{Sin}\,(\alpha + \beta)}, \quad \text{and } \text{AB} = \frac{\text{P Sin}\,\alpha}{\text{Sin}\,(\alpha + \beta)}$$

But AB = OC

i.e. OC = $\frac{\text{P Sin } \alpha}{\text{Sin } (\alpha+\beta)}$

DETERMINATION OF RESOLVED PARTS OF A FORCE

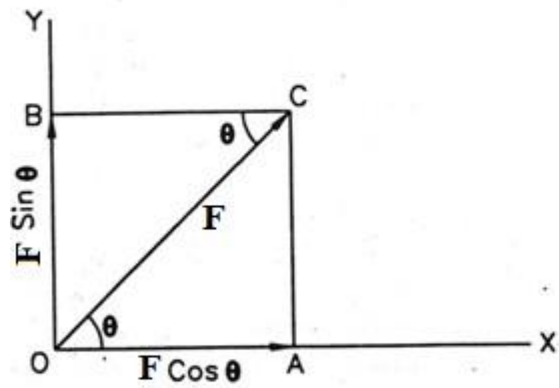


Fig 1.16

Resolved parts of a force mean components of the force along two mutually perpendicular directions.

Let a force F represented in magnitude and direction by OC make an angle θ with OX. Line OY is drawn through O at right angles to OX as shown in figure 1.16.

Through C, lines CA and CB are drawn parallel to OY and OX respectively. Then the resolved parts of the force F along OX and OY are represented in magnitude and direction by OA and OB respectively.

Now in the right angled Δ AOC,

cosθ = OA / OC = OA / F   i.e OA = F cos θ

Since OA is parallel to BC, <OCB = <AOC = θ

In the right angled Δ OBC, sinθ = OB / OC = OB / F i.e, OB = F sin θ

Thus, the resolved parts of F along OX and OY are respectively. F cos θ and F sin θ.

SIGNIFICANCE OF THE RESOLVED PARTS OF A FORCE

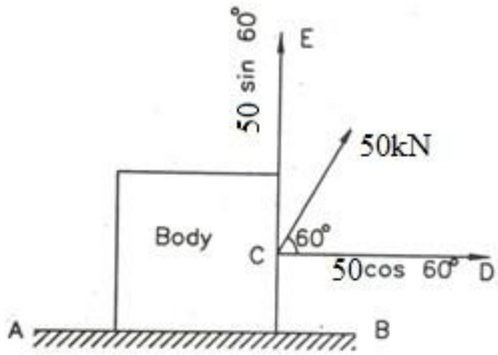


Fig 1.17

Let 50 KN force is required to be applied to a body along a horizontal direction CD in order to move the body along the plane AB. Then it can be said that to move the body along the same plane AB, a force of 50kN is to be applied at an angle of 60° with the horizontal as CD = 50 cos60° = 25kN.

Similarly, if a force of 43.3kN is required to be applied to the body to lift it vertically upward, then the body will be lifted vertically upward if a force of 50kN is applied to the body at an angle of 60° with the horizontal, as the resolved part of 50kN along the vertical CE = 50 sin60° = 43.3kN.

Thus, the resolved part of a force in any direction represents the whole effect of the force in that direction.

1.4 RESULTANT AND COMPONENT

Resultant of two or more forces is a single force whose effect on a body is the same as the given forces taken together acting on the body. In figure 1.20, **R** is the resultant of forces **P** and **Q**.

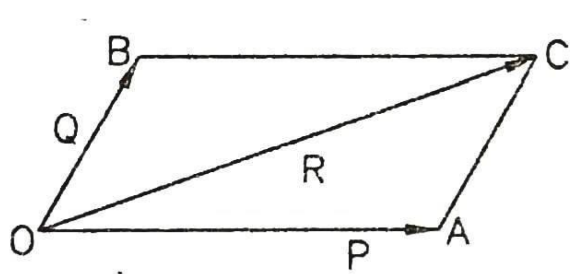


Fig 1.18 Resultant and component

If R is the resultant of two forces P and Q, it means forces P and Q can be replaced by R. Similarly, R can be replaced by two forces P and Q whose joint effect on a body will be the same as R on the body. Then these two forces P and Q are called components of R.

Or we can say:

If a number of forces, P, Q, R ... etc are acting simultaneously on a particle, then it is possible to find out a single force which could replace them i.e., which would produce the same effect as produced by all the given forces. This single force is called resultant force and the given forces P, Q, R ...etc are called component forces.

EQUILIBRIANT

Equilibrant of a system of forces is a single force which will keep the given forces in equilibrium. Evidently, equilibrant is equal and opposite to the resultant of the given forces.

EQUAL FORCES

Two forces are said to be equal when acting on a particle along the same line but in opposite directions, keeping the particle at rest.

METHODS FOR FINDING THE RESULTANT FORCE

Though there are many methods for finding out the resultant force of a number of given forces, yet the following are important from the subject point of view :
1. Analytical method. 2. Method of resolution.

1.4.1 ANALYTICAL METHOD FOR RESULTANT FORCE

The resultant force, of a given system of forces, may be found out analytically by the following methods

1. Parallelogram law of forces. 2. Method of resolution.

PARALLELOGRAM LAW OF FORCES

This theorem states that if two forces acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.
Explanation: Let forces P and Q acting at a point O be represented in magnitude and direction by OA and OB respectively as shown in Fig 1.19. Then, according to the theorem of parallelogram of forces, the diagonal OC drawn through O represents the resultant of P and Q in magnitude and direction.

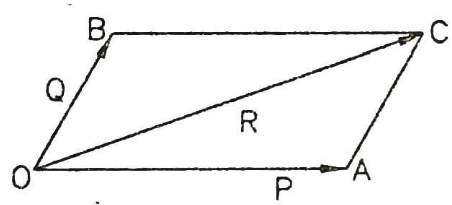


Fig 1.19

DETERMINATION OF THE RESULTANT OF TWO CONCURRENT FORCES WITH THE HELP OF LAW OF PARALLELOGRAM OF FORCES

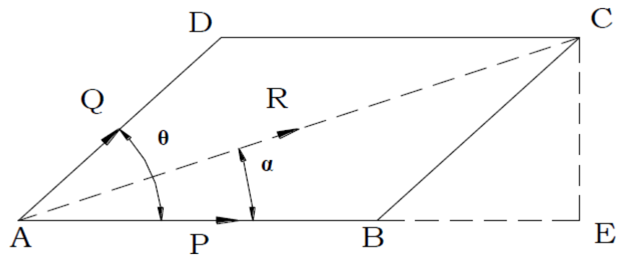


Fig 1.20

Consider, two forces „P" and „Q" acting at and away from point „A" as shown in figure 1.20.

Let, the forces P and Q are represented by the two adjacent sides of a parallelogram AD and AB respectively as shown in fig. Let, θ be the angle between the force P and Q and α be the angle between R and P. Extend line AB and drop perpendicular from point C on the extended line AB to meet at point E.

Consider Right angle triangle ACE,

$AC^2 = AE^2 + CE^2$
$= (AB + BE)^2 + CE^2$
$= AB^2 + BE^2 + 2.AB.BE + CE^2$
$= AB^2 + BE^2 + CE^2 + 2.AB.BE$..............................(1)

Consider right angle triangle BCE,
$BC^2 = BE^2 + CE^2$ and BE = BC.Cos θ

Putting $BC^2 = BE^2 + CE^2$in equation (1), we get

$AC^2 = AB^2 + BC^2 + 2.AB.BE$..................................(2)

Putting BE = BC. Cos θ in equation (2)

$AC^2 = AB^2 + BC^2 + 2.AB.\ BC.\ \cos\theta$

But, AB = P, BC = Q and AC = R

R=$\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

In triangle ACE

$$\tan\alpha = \frac{CE}{AE} = \frac{CE}{AB + BE}$$

But, CE = BC. Sin θ

$$\tan a = \frac{Q \sin \theta}{P + Q \cos \theta}$$

Now let us consider two forcesF_1 and F_2 are represented by the two adjacent sides of a parallelogram

i.e. F_1 and F_2 = Forces whose resultant is required to be found out,
θ = Angle between the forces F_1 and F_2, and
α = Angle which the resultant force makes with one of the forces (say F_1).

Then resultant force

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

And

$$\tan\alpha = \frac{F_2 \sin\theta}{F_1 + F_2\cos\theta}$$

If (α) is the angle which the resultant force makes with the other force F_2, then

$$\tan\alpha = \frac{F_1 \sin\theta}{F_2 + F_1\cos\theta}$$

CASES:

1. If θ = 0 *i.e.*, when the forces act along the same line, then

$$R_{max} = F1 + F2$$

2. If θ = 90° *i.e.*, when the forces act at right angle, then

$$R = \sqrt{F_1^2 + F_2^2}$$

3. If θ = 180° *i.e.*, when the forces act along the same straight line but in opposite directions,then

$$R_{min} = F_1 - F_2$$

In this case, the resultant force will act in the direction of the greater force.

4. If the two forces are equal *i.e.*, when $F_1 = F_2 = F$ then

$$R = \sqrt{F^2 + F^2 + 2F^2\cos\theta} = \sqrt{2F^2(1+\cos\theta)}$$

$$= \sqrt{2F^2 \times 2\cos^2\left(\frac{\theta}{2}\right)} \qquad \ldots\left[\because\ 1+\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)\right]$$

$$= \sqrt{4F^2\cos^2\left(\frac{\theta}{2}\right)} = 2F\cos\left(\frac{\theta}{2}\right)$$

Example 1.1Two forces of 100 N and 150 N are acting simultaneously at a point. What isthe resultant of these two forces, if the angle between them is 45°?

Solution. Given: First force (F_1) = 100 N; Second force (F_2) = 150 N and angle betweenF_1 and F_2 (θ) = 45^0

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos\theta}$$

$$= \sqrt{(100)^2 + (150)^2 + 2 \times 100 \times 150 \cos 45^\circ} \text{ N}$$

$$= \sqrt{10\,000 + 22\,500 + (30\,000 \times 0.707)} \text{ N}$$

$$= 232 \text{ N} \quad \textbf{Ans.}$$

Example 1.2Find the magnitude of the two forces, such that if they act at right angles, theirresultant is$\sqrt{10}$ N. But if they Act at 60°, their resultant is $\sqrt{13}$N.

Solution: Given: Two forces = F_1 and F_2.
First of all, consider the two forces acting at right angles. We know that when the angle betweenthe two given forces is 90°, then the resultant force (R)

$$\sqrt{10} = \sqrt{F_1^2 + F_2^2}$$

$$10 = F_1^2 + F_2^2 \quad \text{...(Squaring both sides)}$$

Similarly, when the angle between the two forces is 60°, then the resultant force (R)

$$\sqrt{13} = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos 60^\circ}$$

$$13 = F_1^2 + F_2^2 + 2F_1 F_2 \times 0.5 \quad \text{...(Squaring both sides)}$$

$$F_1 F_2 = 13 - 10 = 3 \quad \text{...(Substituting } F_1^2 + F_2^2 = 10)$$

We know that $(F_1 + F_{2]})^2 = F_1^2 + F_2^2 + 2F_1F_2 = 10 + 6 = 16$

$$\therefore \quad F_1 + F_2 = \sqrt{16} = 4$$

Similarly $(F_1 - F_2)^2 = F_1^2 + F_2^2 - 2F_1F_2 = 10 - 6 = 4$

$$\therefore \quad F_1 - F_2 = \sqrt{4} = 2$$

Solving equations (*i*) and (*ii*),

$$F_1 = 3 \text{ N} \quad \text{and} \quad F_2 = 1 \text{ N} \quad \textbf{Ans.}$$

DIFFERENCE BETWEEN COMPONENTS AND RESOLVED PARTS

1. When a force is resolved into two parts along two mutually perpendicular directions, the parts along those directions are called resolved parts. But when a force is split into two parts along two assigned directions not at right angles to each other, those parts are called components of the force.

2. All resolved parts are components, but all components are not resolved parts.

3. The resolved part of force in a given direction represents the whole effect of the force in that direction. But the component of a force in a given direction does not represent the whole effect of the force in that direction.

Note:The algebraic sum of the resolved parts of two concurrent forces along any direction is equal to the resolved part of their resultant along the same direction.

ANALYTICAL METHOD OF DETERMINING THE RESULTANT OF ANY NUMBER OF CO-PLANAR CONCURRENT FORCES

Let P, Q, T....... be a number of forces acting at a point O and let R be the required resultant of the given forces.

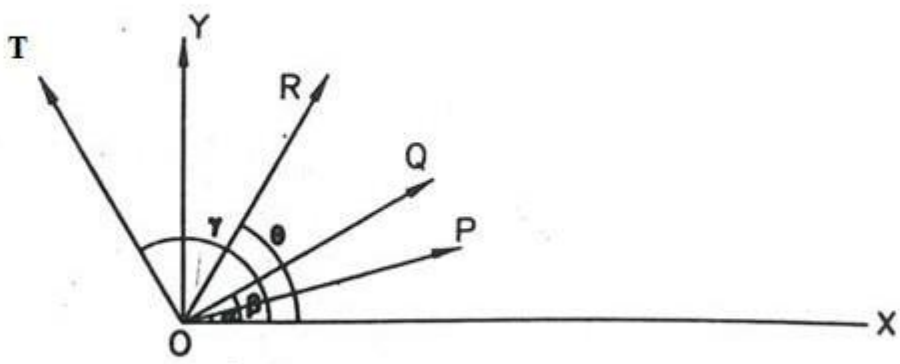


Fig 1.21

Through O, lines OX and OY are drawn at right angles to each other.

Let forces P, Q, T,...... make angles α,β,γ,...... with OX measured in the anticlockwise direction as shown in Fig. Also, let θ = angle made by the line of action of R with OX.

Now, the resolved parts P, Q, T....... along OX are respectively $P\cos\alpha$, $Q\cos\beta$, $T\cos\gamma$ and along OY are respectively $P\sin\alpha$, $Q\sin\beta$, $T\sin\gamma$

Let $\Sigma H = \Sigma X$ = algebraic sum of the resolved parts of the above forces along OX (horizontally)

$\Sigma V = \Sigma Y$= algebraic sum of the resolved parts of the same forces along OY (vertically)

Then, $\Sigma X = P\cos\alpha + Q\cos\beta + T\cos\gamma$.........

$\Sigma Y = P\sin\alpha + Q\sin\beta + T\sin\gamma$......

Now, the resolved parts of R along OX and OY are respectively $R\cos\theta$ and $R\sin\theta$.

$\Sigma X = R\cos\theta$, and $\Sigma Y = R\sin\theta$

$(\Sigma X)^2 + (\Sigma Y)^2 = R^2\cos^2\theta + R^2\sin^2\theta$

$= R^2(\cos^2\theta + \sin^2\theta)$

$= R^2$

$$R = \sqrt{(\sum X)^2 + (\sum Y)^2}$$

$$\frac{R\sin\theta}{R\cos\theta} = \frac{\sum Y}{\sum X} \left(= \frac{\sum V}{\sum H}\right)$$

$$\tan\theta = \frac{\sum Y}{\sum X}\left(= \frac{\sum V}{\sum H}\right)$$

From the above formula, θ can be found out.

Note. When ΣX is +ve, R will lie either in between θ = 0° to 90° or between 270° to 360°.

When ΣX is -ve, R will lie in between 90° to 270°.

When ΣY is +ve, R will lie in between 0 = 0° to 180°.

When ΣY is -ve, R will lie in between 180° to 360°.

Example1.3A particle is acted on by three forces 2, 2√2 and 1 kN. The first force is horizontal and towards the right, the second acts at 45° to the horizontal and inclined right upward, and the third is vertical. Determine the resultant of the given forces.

Solution. See Figure. Let R = required resultant of the given forces.

Then, R= $\sqrt{(\sum X)^2 + (\sum Y)^2}$, where

ΣX = algebraic sum of the resolved part of the given forces along horizontal direction OX, and

ΣY = algebraic sum of the resolved parts of the given forces along vertical direction OY.

Now, ΣX = 2 cos 0° + 2√2 cos 45° + 1 cos 90°

$= 2+2\sqrt{2} \times \frac{1}{\sqrt{2}} + 0 = 4\text{kN}$

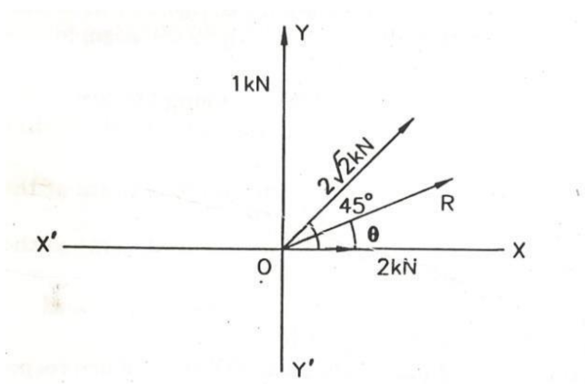


Fig 1.22

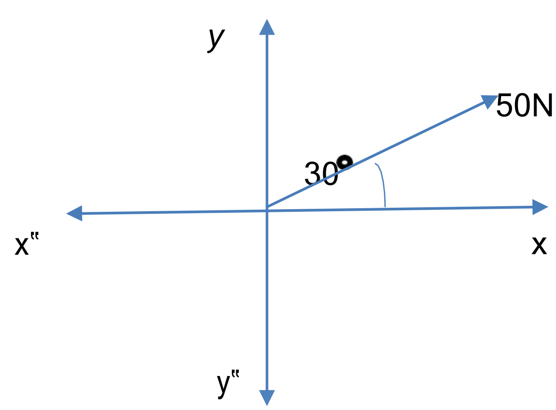
$\Sigma Y = 2 \sin 0° + 2\sqrt{2} \sin 45° + 1 \sin 90°$

$= 0 + 2\sqrt{2}\ x \frac{1}{\sqrt{2}} + 1 = 3kN$

$R = \sqrt{4^2 + 3^2} = 5$ kN

$\tan\theta = \frac{\Sigma Y}{\Sigma X} = \frac{3}{4} = 0.75 \Rightarrow \theta = \tan^{-1} 0.75 = 36.9°$

Example1.4. To resolve the given force into two perpendicular co-ordinates.



Solution:

According to resolution of forces:

We know that $x=50\times\cos 30$, $x=50 \times 0.866$, $x=43.30$N

fig 1.23

$y=50\times\sin 30$, $y=50\times\frac{1}{2}$ $y=25$N

Example 1.5 A triangle ABC has its side AB = 40 mm along positive x-axis and sideBC = 30 mm along positive y-axis. Three forces of 40 N, 50 N and 30 N act along the sides AB, BCand CA respectively. Determine magnitude of the resultant of such a system of forces. Solution. The system of given forces is shown in Fig 1.24.

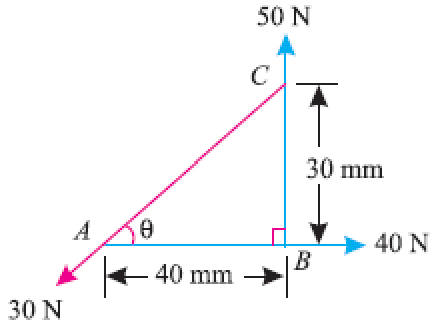


Fig 1.24

From the geometry of the figure, we find that the triangle ABC is a right-angled triangle, in which side AC = 50 mm.

Therefore

$\sin\theta = \frac{30}{50} = 0.6$

$\cos\theta = \frac{40}{50} = 0.8$

Resolving all the forces horizontally (i.e., along AB),

$\sum H = 40 - 30 \cos\theta$

$= 40 - (30 \times 0.8) = 16$ N

and now resolving all the forces vertically (i.e., along BC)

$\sum V = 50 - 30 \sin\theta$

$= 50 - (30 \times 0.6) = 32$ N

We know that magnitude of the resultant force,

$R = \sqrt{(\sum H)^2 + (\sum V)^2} = \sqrt{16^2 + 32^2} = 35.8$ N Ans.

Example 1.6A system of forces are acting at the corners of a rectangular block as shown in Fig 1.25. Determine the magnitude and direction of the resultant force.

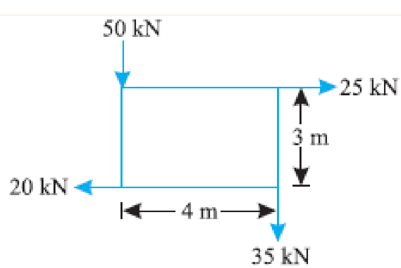


Fig 1.25

Solution.Given:
Let θ = Angle which the resultant force makes with the horizontal
System of forces
Magnitude of the resultant force
Resolving forces horizontally,
$\sum H = 25 - 20 = 5$ kN
and now resolving the forces vertically
$\sum V = (-50) + (-35) = -85$ kN
∴ Magnitude of the resultant force

$R = \sqrt{(\sum H)^2 + (\sum V)^2} = \sqrt{(5)^2 + (-85)^2} = 85.15$ kN Ans.

Since the side AB is along x-axis, and the side BC is along y-axis, therefore it is a right-angled triangle.

Now in triangle ABC,

$AC = \sqrt{AB^2 + BC^2} = \sqrt{40)^2 + (30)^2} = 50$ m

Direction of the resultant force

We know that

$\tan\theta = \frac{\sum V}{\sum H} = \frac{-85}{5} = -17$ or $\theta = 86.6°$

Since ∑H is positive and ∑V is negative, therefore resultant lies between 270° and 360°.
Thusactual angle of the resultant force
= 360° – 86.6° = 273.4° Ans.

Example 1.7. The following forces act at a point :
(i) 20 N inclined at 30° towards North of East,
(ii) 25 N towards North,
(iii) 30 N towards North West, and
(iv) 35 N inclined at 40° towards South of West.
Find the magnitude and direction of the resultant force.
Solution. The system of given forces is shown in Fig 1.26

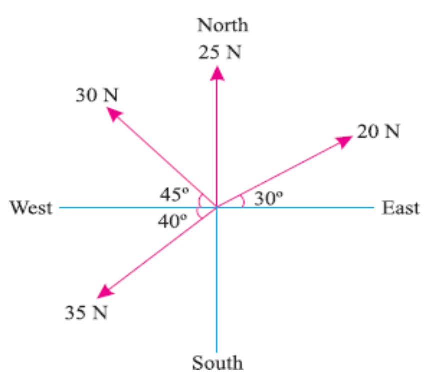


Fig. 1.26

Magnitude of the resultant force

Resolving all the forces horizontally *i.e.*, along East-West line,

ΣH = 20 cos 30° + 25 cos 90° + 30 cos 135° + 35 cos 220° N
= (20 × 0.866) + (25 × 0) + 30 (– 0.707) + 35 (– 0.766) N
= – 30.7 N ...(*i*)
and now resolving all the forces vertically *i.e.*, along North-South line,
ΣV = 20 sin 30° + 25 sin 90° + 30 sin 135° + 35 sin 220° N
= (20 × 0.5) + (25 × 1.0) + (30 × 0.707) + 35 (– 0.6428) N
= 33.7 N ...(*ii*)
We know that magnitude of the resultant force,
$R = \sqrt{(ZH)^2 + (ZV)^2} = \sqrt{(-30.7)^2 + (33.7)^2}$ = 45.6 N **Ans.**
Direction of the resultant force
Let θ = Angle, which the resultant force makes with the East.
We know that,

$\tan\theta = \Sigma V/\Sigma H = 33.7/-30.7 = -1.098$ or $\theta = 47.7°$

Since ΣH is negative and ΣV is positive, therefore resultant lies between 90° and 180°. Thus actual angle of the resultant = 180° – 47.7° = 132.3° **Ans.**

Example 1.8Forces 3, 12$\sqrt{2}$ and 3$\sqrt{2}$kN act at a point towards the East, North-East, and South-West respectively. Determine the resultant of the given forces.

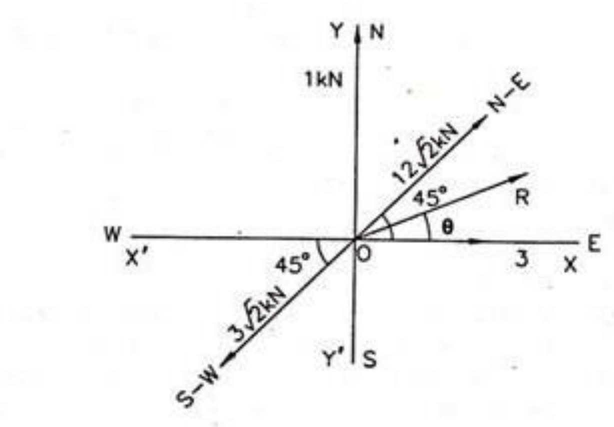


Fig. 1.27

Let
$\sum F_x$=algebraic sum of the resolved parts of the forces along X-axis, and
$\sum F_y$=algebraic sum of the resolved parts of the forces along Y-axis.

$\sum F_x$=3 cos0^0 + 12$\sqrt{2}$ cos 45^0- 3$\sqrt{2}$ cos 45^0
=12kN

$\sum F_y$= 3 sin 0^0 + 12$\sqrt{2}$ sin 45^0 -3$\sqrt{2}$sin 45^0
=9kN
$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$=15kN

Let,
R required resultant of the given forces making an angle**α**with x-axis

$\tan a = \frac{\sum F_y}{\sum F_x} => a = 36.90^0$

1.4.2 GRAPHICAL METHOD

TRIANGLE LAW OF FORCES

It states, "If two forces acting simultaneously on a particle be represented in magnitude and direction by the two sides of a triangle, taken in order; their resultant may be represented in magnitude and direction by the third side of the triangle, taken in opposite order."

Explanation. Let two forces P and Q acting at O be such that they can be represented in magnitude and direction by the sides AB and BC of the triangle ABC. Then, according to the theorem of triangle of forces, their resultant R will be represented in magnitude and direction by AC which is the third side of the triangle ABC taken in the reverse order of CA.

Proof.

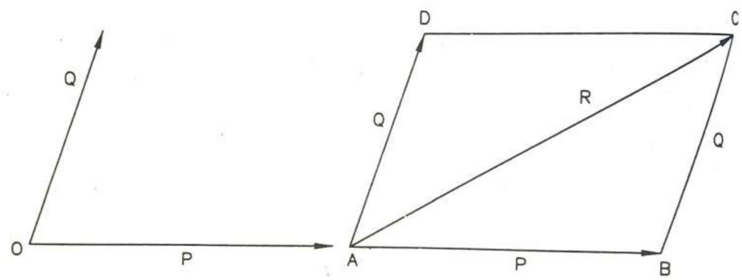


Fig. 1.28

In Fig.1.28 The parallelogram ABCD is completed with sides AB and BC of the triangle ABC. Side AD is equal and parallel to BC. So, force Q is also represented in magnitude and direction by AD. Now, the resultant of P (represented by AB) and Q (represented by AD) is represented in magnitude and direction by the diagonal AC of the parallelogram ABCD. Thus, the resultant of P and Q is represented in magnitude and direction by the third side AC of the triangle ABC taken in the reverse order.

POLYGON LAW OF FORCES

It is an extension of Triangle Law of Forces for more than two forces, which states, "If a number of forces acting simultaneously on a particle, be represented in magnitude and direction, by the sides of a polygon taken in order then the resultant of all these forces may be represented, in magnitude and direction, by the closing side of the polygon, taken in opposite order."

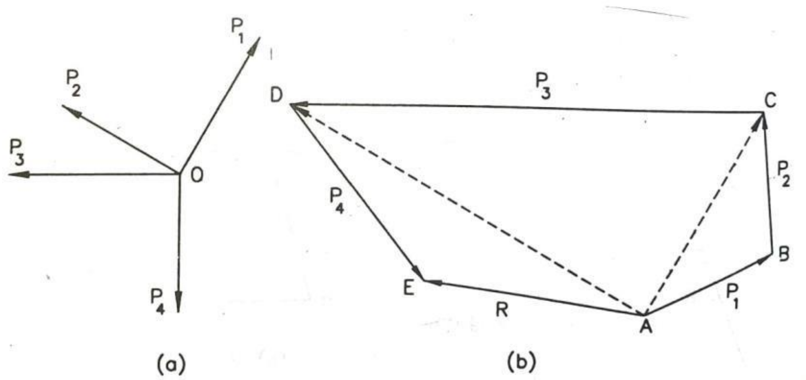


Fig. 1.29

Proof.

Let forces P_1, P_2, P_3 and P_4, acting at a point O be such that they can be represented in magnitude and direction by the sides AB, BC, CD and DE of a polygon ABCDE as shown in fig. 1.29.

We are to prove that the resultant of these forces is represented in magnitude and direction by the side AE in the direction from A towards E.

According to the triangle law of forces, AC represents the resultant R_1 of P_1 and P_2, AD represents the resultant R_2 of R_1 and P_3. Thus, AD represents the resultant of P_1, P_2 and P_3.

According to the same law, AE represents the resultant R_3 of R_2 and P_4. Thus, AE represents the resultant of P_1, P_2, P_3 and P_4.

GRAPHICAL CONDITIONS OF EQUILIBRIUM OF A SYSTEM OF CO-PLANAR CONCURRENT FORCES

The end point of the vector diagram must coincide with the starting point of the diagram. Hence the vector diagram must be a closed figure.

So, graphical condition of equilibrium of a system of co-planar concurrent forces may be stated as follows:

If a system of co-planar concurrent forces be in equilibrium, the vector diagram drawn with the given forces taken in order must be a closed figure.

SPACE DIAGRAM, VECTOR DIAGRAM AND BOW'S NOTATION

Graphical Representation of a Force:

A force can be represented graphically by drawing a straight line to a suitable scale and parallel to the line of action of the given force and an arrowhead indicates the direction.

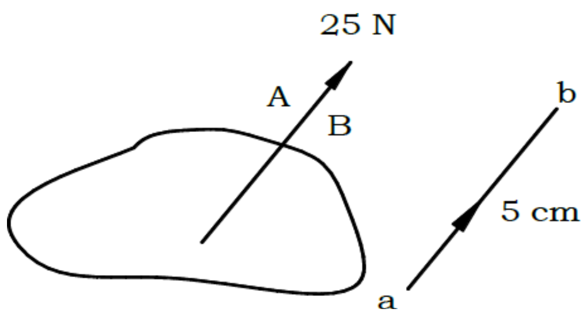


Fig. 1.30

A force in the figure is represented by a vector of length 5 cm (scale 1 cm = 5 N) by drawing a line parallel to the given force and arrowhead indicates the direction of the force.

Space diagram

Space diagram is that diagram which shows the forces in space. In a space diagram the actual directions of forces are marked by straight lines with arrow put on their head to indicate the sense in which the forces act. Following Fig. shows the space diagram of forces P_1, P_2, P_3

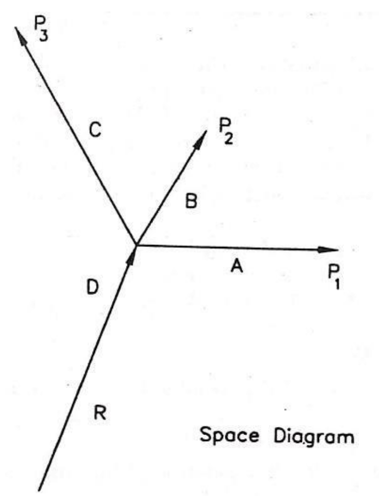


Fig. 1.31

Vector diagram is a diagram which is drawn according to some suitable scale to represent the given forces in magnitude, direction and sense. The resultant of the given forces is represented by the closing line of the diagram and its sense is from the starting point atowards the end point d as shown in Fig 1.32.

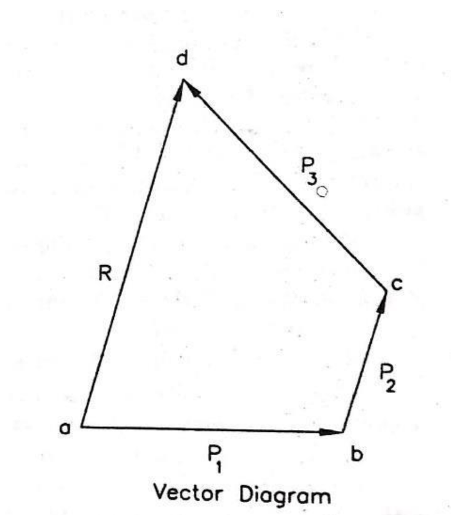


Fig 1.32

Bow’s notationis a method of designating forces in space diagram. According to this system of notation, each force in space diagram is denoted by two capital letters, each being placed on two sides of the line of action of the force. In Fig.1.32, forces P_1 P_2 and P_3 are denoted by AB, BC and CD respectively. In the vector diagram, the corresponding forces are represented by ab, bc and cd respectively. Bow‟s notation is particularly suitable in graphical solution of systems of forces which are in equilibrium.

Example1.9A particle is acted upon by three forces equal to 50 N, 100 N and 130 N, along the three sides of an equilateral triangle, taken in order. Find graphically the magnitude and direction of the resultant force.

Solution. The system of given forces is shown in Fig. First of all, name the forces according to Bow‟s notations as shown in Fig.1.33 a. The 50 Nforces is named as AD, 100 N force as BD and 130 N force as CD

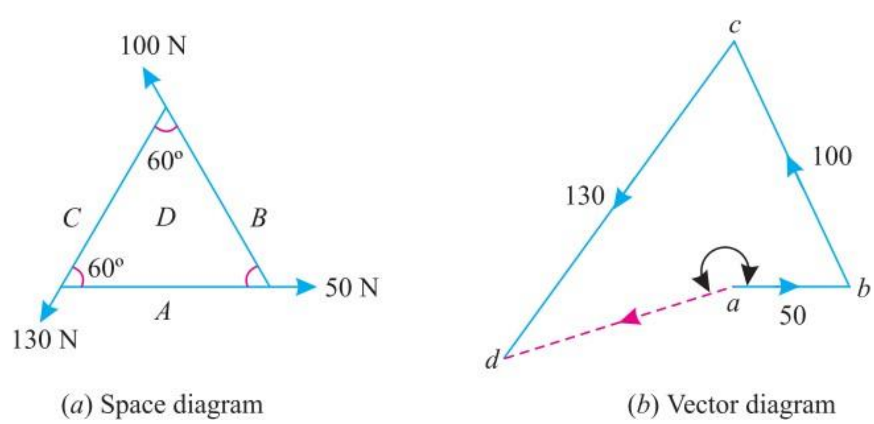


Fig 1.33

Now draw the vector diagram for the given system of forces as shown in Fig 1.33.(b) and as discussed below :

1. Select some suitable point *a*and draw ab equal to 50 N to some suitable scale and parallel to the 50 N force of the space diagram.
2. Through *b*, draw *bc*equal to 100 N to the scale and parallel to the 100 N force of the space diagram.
3. Similarly through *c*, draw *cd*equal to 130 N to the scale and parallel to the 130 N force of the space diagram.
4. Join *ad*, which gives the magnitude as well as direction of the resultant force.
5. By measurement, we find the magnitude of the resultant force is equal to 70 N and acting at an angle of 200° with ab. **Ans.**

CLASSIFICATION OF PARALLEL FORCES

The parallel forces may be, broadly, classified into the following two categories, depending upon their directions:

1. Like parallel forces.

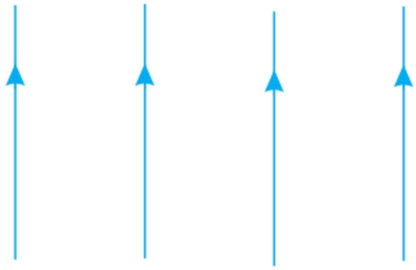
2. unlike parallel forces.

LIKE PARALLEL FORCES

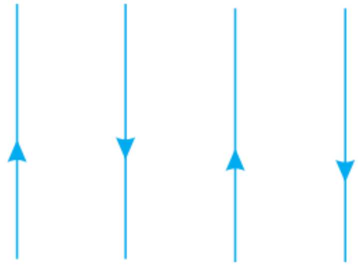
The forces, whose lines of action are parallel to each other and all of them act in the same direction as shown in Fig. 1.34 (*a*) are known as like parallel forces

UNLIKE PARALLEL FORCES

The forces, whose lines of action are parallel to each other and all of them do not act in the same direction as shown in Fig.1.34 (*b*) are known as unlike parallel forces.



(*a*) Like parallel forces



(*b*) Unlike parallel forces.

Fig 1.34

The magnitude and position of the resultant force, of a given system of parallel forces (like or unlike) may be found out analytically or graphically

1.4.3 ANALYTICAL METHOD OF DETERMINATION OF THE RESULTANT OF A SYSTEM OF LIKE AND UNLIKE PARALLEL FORCES

In this method, the sum of clockwise moments is equated with the sum of anticlockwise momentsabout a point.

ANALYTICAL METHOD OF DETERMINING THE POINT OF APPLICATION OF THE RESULTANT OF A SYSTEM OF LIKE AND UNLIKE NON CONCURRENT PARALLEL FORCES

We know that the algebraic sum of the moments of any number of co-planar forces (concurrent or non-concurrent) about any point in their plane is equal to the moment of their resultant about the same point. This principle is applied in determining the point of application of the resultant of any number of parallel forces.

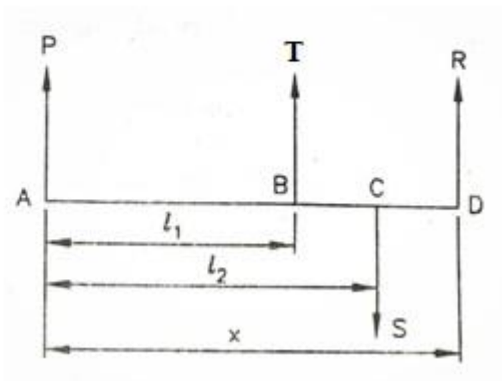


Fig 1.35

Let parallel forces P, T and S be acting at the points A, B and C respectively as shown in Fig 1.35.

The resultant of the parallel forces is given by R= P+T-S.

Let x = required distance of the point of application of R from A

i.e. x = AD.

Taking moments about A, we get

$R \times x - S \times l_2 + T \times l_1 = 0$

$R \times x = S \times l_2 - T \times l_1$

$$x = \frac{S \times l_2 - T \times l_1}{R} = \frac{S \times l_2 - T \times l_1}{P + Q - S}$$

EXAMPLE 1.10A beam 3 m long weighing 400 N is suspended in a horizontal position by two vertical strings, each of which can withstand a maximum tension of 350 N only. How far a body of 200 N weight be placed on the beam, so that one of the strings may just break ?

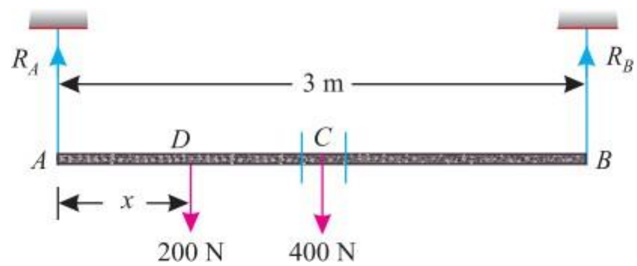


Fig 1.36

Let *x* = Distance between the body of weight 200 N and support *A*.
We know that one of the string (say *A*) will just break, when the tension will be 350 N. (*i.e.*, R_A = 350 N).
Now taking clockwise and anticlockwise moments about *B* and equating the same,
350 × 3 = 200 (3 – *x*) + 400 × 1.5
1 050 = 600 – 200 *x* + 600 = 1200 – 200 *x*
200 *x* = 1 200 – 1 050 = 150

X = $\frac{150}{200} = 0.75m$

Example 1.11. Two unlike parallel forces of magnitude 400 N and 100 N are acting in such away that their lines of action are 150 mm apart. Determine the magnitude of the resultant force andthe point at which it acts.
Solution. Given : The system of given force is shown in Fig

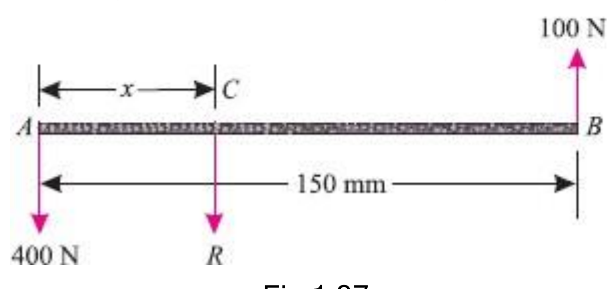


Fig 1.37

Magnitude of the resultant force
Since the given forces are unlike and parallel, therefore magnitude of the resultant force,
R = 400 – 100 = 300 N **Ans.**

Point where the resultant force acts

Let x = Distance between the lines of action of the resultant force and Ain mm.

Now taking clockwise and anticlockwise moments about A and equating the same,

$300 \times x = 100 \times 150 = 15\,000$

$x = 15000/300 = 50$mm ans.

GRAPHICAL METHOD FOR THE RESULTANT OF PARALLEL FORCES

Consider a number of parallel forces (say three like parallel forces) P_1, P_2 and P_3whoseresultant is required to be found out as shown in Fig 1.38.a

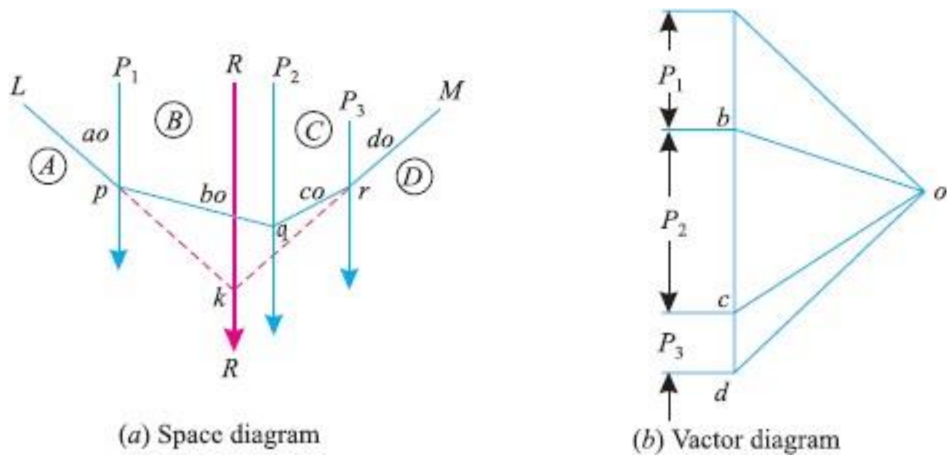


Fig 1.38

First of all, draw the space diagram of the given system of forces and name them according to Bow‟s notations as shown in Fig.1.38 (*a*). Now draw the vector diagram for the given forces as shown in Fig.1.38 (*b*) and as discussed below :

1. Select some suitable point a, and draw ab equal to the force AB (P_1) and parallel to it to some suitable scale.
2. Similarly draw bc and cd equal to and parallel to the forces BC (P_2) and CD (P_3) respectively.
3. Now take some convenient point o and joint oa, ob, oc and od.
4. Select some point p, on the line of action of the force AB of the space diagram and through it draw a line Lpparallel to ao. Now through p draw pq parallel to bo meeting the line of action of the force BC at q.
5. Similarly draw qrand rM parallel to co and do respectively.
6. Now extend Lp and Mr to meet at k. Through k, draw a line parallel to ad, which gives the required position of the resultant force.
7. The magnitude of the resultant force is given by ad to the scale.

Note. This method for the position of the resultant force may also be used for any system of forces i.e. parallel, like, unlike or even inclined.

1.5 MOMENT OF A FORCE

It is the turning effect produced by a force, on the body, on which it acts. The moment of a force is equal to the product of the force and the perpendicular distance of the point, about which the moment is required and the line of action of the force.

Mathematically, moment,

$M = P \times l$

where P = Force acting on the body,

andl = Perpendicular distance between the point, about which the moment is required and the line of action of the force.

Moment of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.

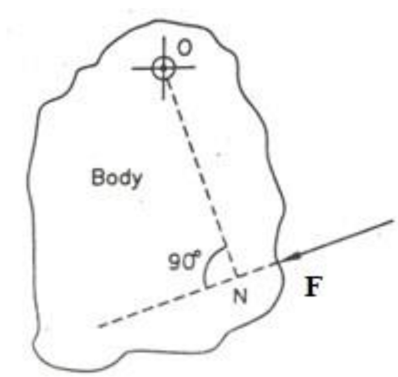


Fig 1.39

Let a force P act on a body which is hinged at O.

Then, moment of P about the point O in the body is = F x ON,

where :ON = perpendicular distance of O from the line of action of the force F.

MOMENT OF A FORCE ABOUT AN AXIS

Let us consider a door leaf hinged to a vertical wall by several hinges. Let us consider a vertical axis XY passing through hinges as shown in Fig 1.40.

Let a force F be applied to the door leaf at right angles to its plane and at a perpendicular distance of *l* from the XY-axis. Then, moment of the force F about XY-axis = F x *l*.

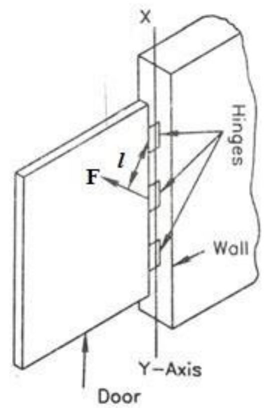


Fig 1.40

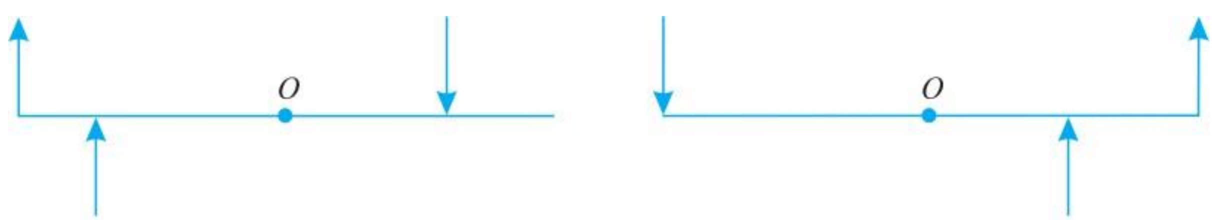
UNIT OF MOMENT

Unit of moment depends upon unit of force and unit of length.

If, however, force is measured in Newton and distance is measured in meter, the unit of moment will be Newton meter (Nm). If force is measured in kilo Newton and distance is measured in meter, unit of moment will be kilo Newton meter (kNm) and so on. Unit of moment is the same as that of work. But work is completely different from moment.

TYPES OF MOMENTS

Broadly speaking, the moments are of the following two types:
1. Clockwise moments. 2. Anticlockwise moments.



(*a*) Clockwise moments (*b*) Anticlockwise moments

Fig 1.41

Clockwise moment is the moment of a force, whose effect is to turn or rotate the body, about the point in the same direction in which hands of a clock move as shown in Fig. 1.41(*a*) .

Anticlockwise moment is the moment of a force, whose effect is to turn or rotate the body, about the point in the opposite direction in which the hands of a clock move as shown in Fig1.41 (*b*).

POSITIVE MOMENT AND NEGATIVE MOMENT

It is found that some moments acting on a body have a tendency to turn the body in the clockwise direction and some other moments acting „on the same body have a tendency to turn the body in the anti-clockwise or counter clockwise direction.

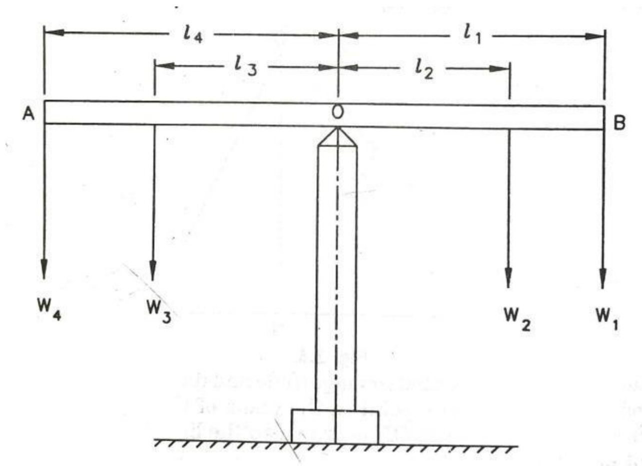


Fig 1.42

In order to distinguish turning tendency in the clockwise direction from that in the anti-clockwise direction, it has become necessary to treat moment in one direction as positive and moment in the reverse direction as negative. Usually, anti-clockwise moment is taken as positive moment and clockwise moment is taken as negative moment. But there is no hard and fast rule regarding sign convention of moments.

ALGEBRAIC SUM OF THE MOMENTS

With reference to Fig1.42, a bar AB is held in position on a pivot O under the action of four loads W_1, W_2, W_3 and W_4, whose lines of action are at perpendicular distances of l_1, l_2, l_3, $l_4$respectively from O. Then, moment of about O = $W_1 \times l_1$. This moment has a tendency to turn the bar about O in a vertical plane in the clockwise direction. The moment due to W_2 about O = $W_2 \times l_2$. This moment also has a tendency to turn the bar AB in the clockwise direction in a vertical plane about O.

The moment due to W_3 about O = $W_3 x l_3$. This moment has a tendency to turn the bar AB in the anti-clockwise direction in a vertical plane about O. The moment due to W, about O = $W_4 \times l_4$. This moment also has a tendency to turn the bar AB in the anti-clockwise direction in the vertical plane about O.

Algebraic sum means summation considering proper signs of the physical quantities. Hence, algebraic sum of the moments of W_1, W_2, W_3, W_4 about O = $W_3 \times l_3 + W_4 \times l_4 - W_1 \times l_1 - W_2 \times l_2$

GEOMETRICAL REPRESENTATION OF THE MOMENT OF THE FORCE ABOUT A POINT

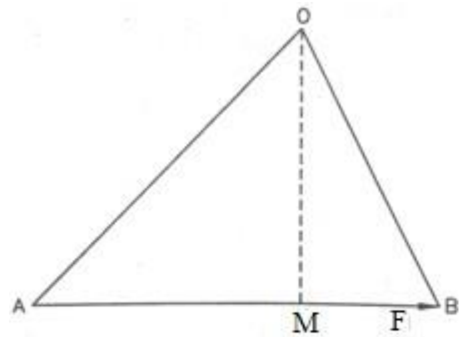


Fig 1.43

Let a force F represented in magnitude and direction by AB be acting on a body and let O be any point in the plane of the force F as shown In Fig 1.43.

From O, perpendicular OM is drawn on the line of action of F. Then, moment of F about

O=FxOM=2x$\frac{1}{2}F$ x OM = 2 x $\frac{1}{2}$AB x OM = 2x Area of ΔAOB.

Thus, the moment of a force about a point is represented by twice the area of the triangle formed by joining the point to the extremities of the straight line which represents the force.

VARIGNON'S THEOREM

Varignon"s theorem states that the algebraic sum of the moment, two forces about any point in their plane is equal to the moment of the, resultant about the same point.

Proof.

Case (i) When the forces are concurrent

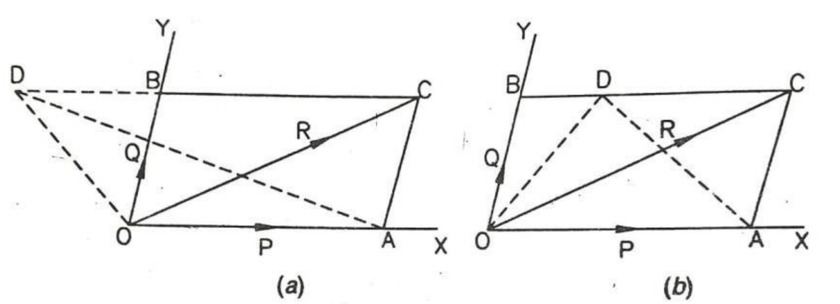


Fig 1.44

Let P and Q be any two forces acting at a point O along lines OX and OY respectively and let D be any point in their plane as shown in Fig 1.44.

Line DC is drawn parallel to OX to meet OY at B. Let in some suitable scale, line OB represent the force Q in magnitude and direction and let in the same scale, OA represent the force P in magnitude and direction.

With OA and OB as the adjacent sides, parallelogram OACB is completed and OC is joined. Let R be the resultant of forces P and Q. Then, according to the “Theorem of parallelogram of forces”, R is represented in magnitude and direction by the diagonal OC of the parallelogram OACB.

The point D is joined with points O and A. The moments of P, Q and R about D are given by 2 x area of ΔAOD, 2 x area of ΔOBD and 2 x area of ΔOCD respectively.

With reference to Fig1.44(a), the point D is outside the <AOB and the moments of P, Q and R about D are all anti-clockwise and hence these moments are treated as +ve.

Now, the algebraic sum of the moments of P and Q about

D = 2ΔAOD + 2ΔOBD

= 2 (ΔAOD + ΔOBD)

= 2 (ΔAOC + ΔOBD) {See note below]

= 2 (ΔOBC + ΔOBD)

= 2ΔOCD = Moment of R about D.

[Note. As AOC and AOD are on the same base and have the same altitude. ΔAOD = ΔOBC. .

Again, As AOC and OBC have equal bases and equal altitudes. ΔAOC = ΔOBC].

With reference to Fig 1.44 (b), the point D is within the <AOB and the moments of P, Q and R about D are respectively anti-clockwise, clockwise and anti-clockwise.

Now, the algebraic sum of the forces P and Q about

D = 2ΔAOD-2 ΔOBD = 2 (ΔAOD-ΔOBD)= 2 (ΔAOC- ΔOBD)=2(ΔOBC - ΔOBD)

= 2ΔOCD = Moment of R about D

Case (ii) : When the forces are parallel

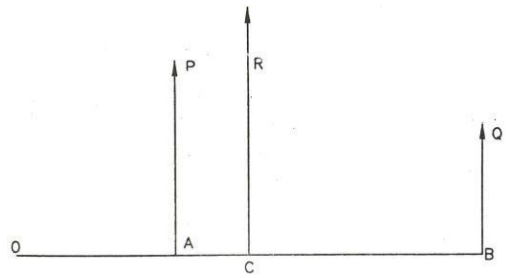


Fig 1.45

Let P and Q be any two like parallel forces (i.e. the parallel forces whose lines of action are parallel and which act in the same sense) and O be any point in their plane.

Let R be the resultant of P and Q.

Then, R=P+Q

From O, line OACB is drawn perpendicular to the lines of action of forces P, Q and R intersecting them at A, B and C respectively as shown in Fig 1.45.

Now, algebraic sum of the moments of P and Q about O

= Px OA + Q x OB

= P x (OC - AC) + Q x (OC + BC)

=PxOC - PxAC+QxOC+Qx BC.

But P x AC = Q x BC

Algebraic sum of the moments of P and Q about O

=PxOC+QxOC

=(P+Q)xOC=RxOC = Moment of R about O.

In case of unlike parallel forces also it can be proved that the algebraic sum of the moments of two unlike parallel forces (i.e. the forces whose lines of action are parallel but which act in reverse senses) about any point in their plane is equal to the moment of their resultant about the same point.

PRINCIPLE OF MOMENTS

1. If a system of co-planar forces (concurrent or non-concurrent) is in equilibrium, the algebraic sum of the moments of those forces about any point in their plane is zero, i.e., the sum of the clockwise moments about any point in their plane is equal to the sum of the anticlockwise moments about the same point.

2. The algebraic sum of the moments of any number of co-planar forces (concurrent or non-concurrent) about a point lying on the line of action of their resultant is zero.

3. From 1 and 2 above, it can be concluded that if the algebraic sum of the moments of any number of co-planar forces about any point in their plane is zero, either the forces are in equilibrium or their resultant passes through that point.

Example 1.12A force of 15 N is applied perpendicular to the edge of a door 0.8 m wide as shown in Fig (a). Find the moment of the force about the hinge. If this force is applied at an angle of 60° to the edge of the same door, as shown in Fig.1.47 (b), find the moment of this force.

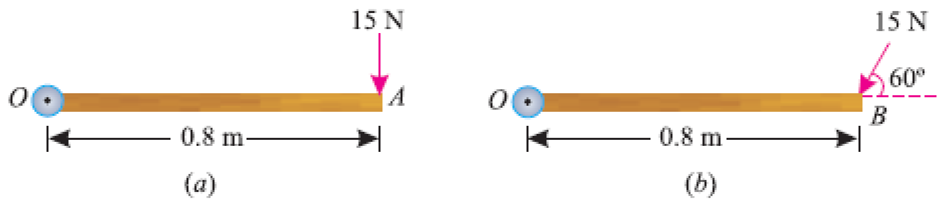


Fig 1.46

Solution. Given : Force applied (P) = 15 N and width of the door (l) = 0.8 m
Moment when the force acts perpendicular to the door
We know that the moment of the force about the hinge,
$= P \times l = 15 \times 0.8 = 12.0$ N-m **Ans.**
Moment when the force acts at an angle of 60° to the door
This part of the example may be solved either by finding out the perpendicular distance betweenthe hinge and the line of action of the force as shown in Fig 1.47(a) or by finding out the verticalcomponent of the force as shown in Fig 1.47.(b).

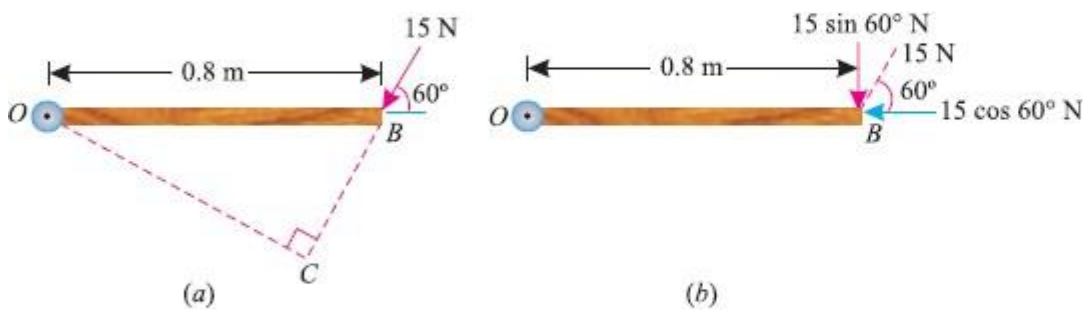


Fig 1.47

From the geometry of Fig.1.47(*a*), we find that the perpendicular distance between the line ofaction of the force and hinge,
$OC = OB \sin 60° = 0.8 \times 0.866 = 0.693$ m
∴ Moment $= 15 \times 0.693 = 10.4$ N-m **Ans.**
In the second case, we know that the vertical component of the force
$= 15 \sin 60° = 15 \times 0.866 = 13.0$ N
∴ Moment $= 13 \times 0.8 = 10.4$ N-m **Ans.**
Note. Since distance between the horizontal component of force (15 cos 60°) and the hingeis zero, therefore moment of horizontal component of the force about the hinge is also zero

Example 1.13A uniform plank ABC of weight 30 N and 2 m long is supported at one end Aand at a point B 1.4 m from A as shown in Fig. Find the maximum weight W, that can be placed at C, so that the plank does not topple.

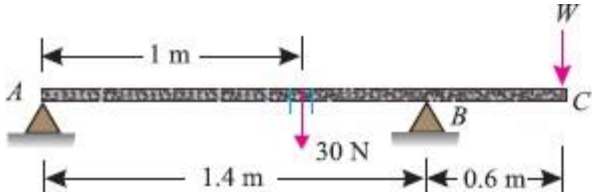


Fig 1.48

Solution. Weight of the plank ABC = 30 N; Length of the plank ABC = 2 m and distance between end A and a point B on the plank (AB) = 1.4 m.
We know that weight of the plank (30 N) will act at its midpoint, as it is of uniform section.
This point is at a distance of 1 m from A or 0.4 m from B as shown in the figure.
We also know that if the plank is not to topple, then the reaction at A should be zero for the maximum weight at C.
Now taking moments about B and equating the same,
$30 \times 0.4 = W \times 0.6$
$W = 12/0.6 = 20N$ ANS.

EXAMPLE 1.14 A uniform wheel of 600 mm diameter, weighing 5 kN rests against a rigid rectangular block of 150 mm height.

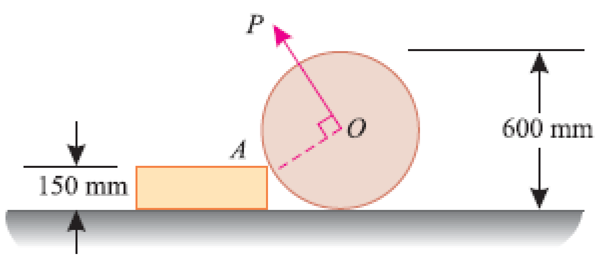


Fig 1.49

Find the least pull, through the centre of the wheel, required just to turn the wheel over thecorner A of the block. Also find the reaction on the block. Take all the surfaces to be smooth.

Solution. Given : Diameter of wheel = 600 mm; Weight of wheel = 5 kN and height of the block = 150 mm.
Least pull required just to turn the wheel over the corner
Let P = Least pull required just to turn the wheel in kN.
A little consideration will show that for the least pull, it must be applied normal to AO.

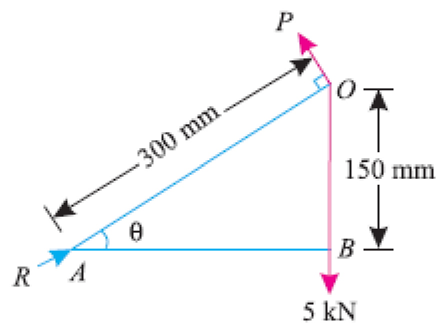


Fig 1.50

From the geometry of the figure, we find that

$\sin\theta = \frac{150}{300} = 0.5$

$\Rightarrow \theta = 30°$

and, $AB = \sqrt{(300)^2 - (150)^2} = 260$ mm

Now taking moments about *A* and equating the same,

$P \times 300 = 5 \times 260 = 1300$

$\therefore P = \frac{1300}{300} = 4.33$ KN

Reaction on the block

Let, R = Reaction on the block in KN

Resolving the forces horizontally and equating the same

$R \cos 30° = P \sin 30°$

$\therefore \quad R = \frac{P\sin 30}{\cos 30} = \frac{4.33 \times 0.5}{0.866} = 2.5$ KN

The position of a resultant force may be found out by moments as discussed below:

1. First of all, find out the magnitude and direction of the resultant force by the method of resolution as discussed earlier in chapter „Composition and Resolution of Forces".

2. Now equate the moment of the resultant force with the algebraic sum of moments of the given system of forces about any point. This may also be found out by equating the sum of clockwise moments and that of the anticlockwise moments about the point, through which the resultant force will pass.

EXAMPLE 1.15.Three forces of 2P, 3P and 4P act along the three sides of an equilateral triangle of side 100 mm taken in order. Find the magnitude and position of the resultant force.

Solution

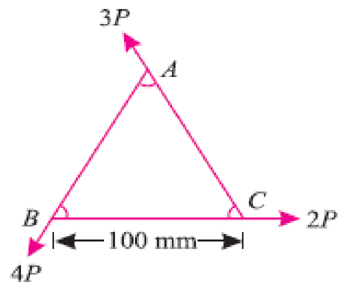


Fig 1.51

Magnitude of the resultant force

Resolving all the forces horizontally,

$\Sigma H = 2P + 3P \cos 120° + 4P \cos 240°$

$= 2P + 3P(-0.5) + 4P(-0.5)$

$= -1.5\,P$ .... (i)

and now resolving all the forces vertically.

$\Sigma V = 3P \sin 60° - 4P \sin 60°$

$= (3P \times 0.866) - (4P \times 0.866)$

$= -0.866\,P$ .... (ii)

We know that magnitude of the resultant force

$R = \sqrt{(\Sigma\ )^2 + (\Sigma\ )^2} = \sqrt{(-1.5P)^2 + (-0.866\,P)^2} = 1.732\,P$

Position of the resultant force

Let x = Perpendicular distance between B and the line of action of the resultantforce.

Now taking moments of the resultant force about B and equating the same,

$1.732\,P \times x = 3P \times 100 \sin 60° = 3P \times (100 \times 0.866) = 259.8\,P$

$\therefore x = \frac{259.8}{1.732} = 150\ \text{mm}$ (The moment of the force 2*P* and 4*P* about the point *B* will be zero, as they passthrough it.)

COUPLE

A pair of two equal and unlike parallel forces (i.e. forces equal in magnitude, with lines of action parallel to each other and acting in opposite directions) is known as a couple.

As a matter of fact, a couple is unable to produceanytranslatory motion (*i.e*., motion in a straight line).But it produces a motion of rotation in the body, onwhich it acts. The simplest example of a couple is theforces applied to the key of a lock, while locking orunlocking it.

ARM OF A COUPLE: The perpendicular distance between the lines of action of the two equal and opposite parallel forces, is known as arm of the couple.

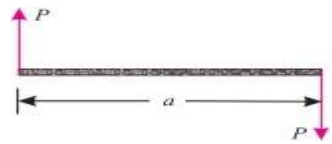


Fig 1.52

MOMENT OF A COUPLE

The moment of a couple is the product of the force (i.e., one of the forces of the two equal and opposite parallel forces) and the arm of the couple. Mathematically:

Moment of a couple = P × a

where P = Magnitude of the force, and a = Arm of the couple.

CLASSIFICATION OF COUPLES The couples may be, broadly, classified into the following two categories, depending upon their direction, in which the couple tends to rotate the body, on which it acts: 1. Clockwise couple, and 2. Anticlockwise couple.

CLOCKWISECOUPLE: A couple, whose tendency is to rotate the body, on which it acts, in a clockwise direction, is known as a clockwise couple as shown in Fig. 1.53 (a). Such a couple is also called positive couple.

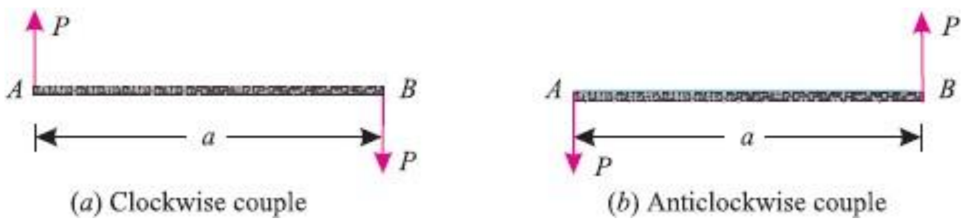


Fig 1.53

ANTICLOCKWISE COUPLE: A couple, whose tendency is to rotate the body, on which it acts, in an anticlockwise direction, is known as an anticlockwise couple as shown in Fig 1.53(b). Such a couple is also called a negative couple.

UNITS OF COUPLE:

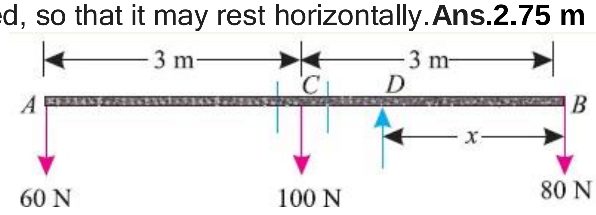
The SI unit of couple will be Newton-meter (briefly written as N-m). Similarly, the units of couple may also be kN-m (i.e. kN × m), N-mm (i.e. N × mm) etc.

CHARACTERISTICSOF A COUPLE: A couple (whether clockwise or anticlockwise) has the followingcharacteristics:

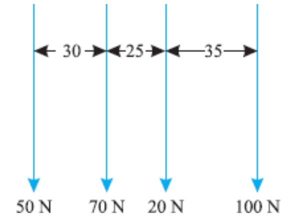
1. The algebraic sum of the forces, constituting the couple, is zero.
2. The algebraic sum of the moments of the forces, constituting the couple, about any point is the same, and equal to the moment of the couple itself.
3. A couple cannot be balanced by a single force. But it can be balanced only by a couple of opposite sense.
4. Any no. of coplanar couples can be reduced to a single couple, whose magnitude will be equal to the algebraic sum of the moments of all the couples.

EXERCISES

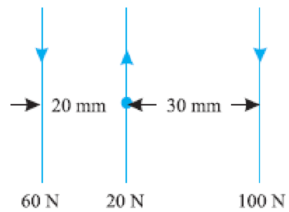
1. Define the term „force‟, and state clearly the effects of force.
2. What are the various characteristics of a force?
3. Distinguish clearly between resolution of forces and composition of forces.
4. What are the methods for finding out the resultant force for a given system of forces?
5. State and prove parallelogram law of forces.
6. State triangle law of forces and polygon law of forces.
7. Show that the algebraic sum of the resolved part of a number of forces in a given direction,is equal to the resolved part of their resultant in the same direction.
8. Explain clearly the procedure for finding out the resultant force analytically as well asgraphically.
9. What is meant by moment of a force? How will you explain it mathematically?
10. How will you represent the moment of a force geometrically?
11. Explain clearly the difference between clockwise moments and anticlockwise moments.
12. State clearly the law of moments.
13. State the Varignon‟s principle of moments.
14. What do you understand by the term „parallel forces‟ ? Discuss their classifications.
15. Distinguish clearly between like forces and unlike forces.
16. What is a couple ? What is the arm of a couple and its moment ?
17. Discuss the classification of couples and explain clearly the difference between a positive couple and a negative couple.
18. State the characteristics of a couple.
19. The forces 20 N, 30 N, 40 N, 50 N and 60 N are acting at one of the angular points of a regular hexagon, towards the other five angular points, taken in order. Find the magnitude and direction of the resultant force.**Ans. 155.8 N, θ = 76.6°**
20. A horizontal line PQRS is 12 m long, where PQ = QR = RS = 4 m. Forces of1000 N, 1500 N, 1000 N and 500 N act at P, Q, R and S respectively with downward direction. Thelines of action of these forces make angles of 90°, 60°, 45° and 30° respectively with PS. Find the magnitude, direction and position of the resultant force**.Ans. 3765 N, θ = 59.8°**
21. The following forces act at a point :
    (i) 20 N inclined at 30° towards North of East.
    (ii) 25 N towards North.
    (iii) 30 N towards North West and
    (iv) 35 N inclined at 40° towards South of West.
    Find the magnitude and direction of the resultant force. **Ans.45.6 N, 132°**
22. Two like parallel forces of 50 N and 100 N act at the ends of a rod 360 mm long. Find the magnitude of the resultant force and the point where it acts.**Ans.240 mm**
23. A uniform beam AB of weight 100 N and 6 m long had two bodies of weights60 N and 80 N suspended from its two ends as shown in Fig.Find analytically at what point the beam should be supported, so that it may rest horizontally.**Ans.2.75 m**



24. Find graphically the resultant of the forces shown in Fig. The distances between the forces are in mm. Also find the point, where the resultant acts.

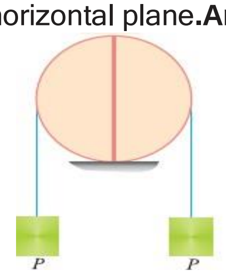


25. Find graphically the resultant of the forces shown in Fig. Also find the point where the resultant force acts.

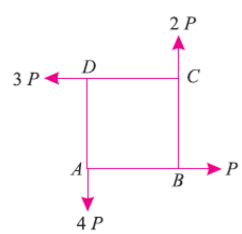


26. A rod AB 2·5 m long is supported at A and B. The rod is carrying a point load of 5 kN ata distance of 1 m from A. What are the reactions at A and B ?**[Ans. 2 kN ; 3 kN]**

27. Two halves of a round homogeneous cylinder are held together by a threadwrapped round the cylinder with two weights each equal to P attached to its ends as shown inFig.The complete cylinder weighs W Newton. The plane of contact, of both of its halves, is vertical.Determine the minimum value of P, for which both halves of the cylinderwill be in equilibrium on a horizontal plane.**Ans. P = 2W/3**π



28. Four forces equal to P, 2P, 3P and 4P are respectively acting along the foursides of square ABCD taken in order. Find the magnitude, direction and position of the resultant force.**Ans.** $2\sqrt{2}P$**,** $\theta = 45°$**,** $5a/2\sqrt{2}$



CHAPTER 2: EQUILIBRIUM OF FORCES

LEARNING OUTCOMES:

On completion of the subject, the student will be able to:

- Understand the concept of equilibrium
- Understand and analyse the various analytical and graphical conditions required for equilibrium
- Understand the concept of free body diagram
- Understand the concept of Lami‟s theorem
- Solve problems using free body diagram and Lami‟s theorem.

2.1 DEFINITION

A little consideration will show, that if the resultant of a number of forces, acting on a particle is zero, the particle will be in equilibrium. Such a set of forces, whose resultant is zero, are called equilibrium forces..

A body can be said to be in **equilibrium**when all the force acting on a body balance each other or in other word there is no net force acting on the body.

Equilibrium of a body is a state in which all the forces acting on the body are balanced (cancelled out), and the net force acting on the body is zero.

i.e $\Sigma F = 0$

PRINCIPLES OF EQUILIBRIUM

1. Two force principle. As per this principle, if a body in equilibrium is acted upon by two forces, then they must be equal, opposite and collinear.
2. Three force principle. As per this principle, if a body in equilibrium is acted upon by three forces, then the resultant of any two forces must be equal, opposite and collinear with the third force.
3. Four force principle. As per this principle, if a body in equilibrium is acted upon by four forces, then the resultant of any two forces must be equal, opposite and collinear with the resultant of the other two forces.

ANALYTICAL CONDITIONS OF EQUILIBRIUM OF A CO-PLANAR SYSTEM OF CONCURRENT FORCES

We know that the resultant of a system of co-planar concurrent forces is given by

$R = \sqrt{(\sum X)^2 + (\sum Y)^2}$, where ΣX (=ΣH) = algebraic sum of the resolved parts of the forces along a horizontal direction, and ΣY (=ΣV) = algebraic sum of the resolved parts of the forces along a vertical direction

Or, $R^2 = (\Sigma X)^2 + (\Sigma Y)^2$

If the forces are in equilibrium, R = 0 =>O = $(\Sigma X)^2 + (\Sigma Y)^2$

Sum of the squares of two quantities is zero when each quantity is separately equal to zero.

i.e. $\Sigma X = 0$, $\Sigma Y = 0$

Hence necessary and sufficient conditions of a system of, co-planar concurrent forces are:

1. The algebraic sum of the resolved parts of the forces in some assigned direction is equal to zero, and

2. The algebraic sum of the resolved parts of the forces in a direction at right angles to the assigned direction is equal to zero.

ANALYTICAL CONDITIONS OF EQUILIBRIUM OF A SYSTEM OF COPLANAR NON-CONCURRENT FORCES

If R = resultant of a system of co-planar non-concurrent forces,
ΣX = algebraic sum of the resolved parts of those forces along any direction, and
ΣY = algebraic sum of the resolved parts of those forces along a direction at right angles to the previous direction.

Then, R= $\sqrt{(\sum X)^2 + (\sum Y)^2}$

$R^2 = (\Sigma X)^2 + (\Sigma Y)^2$

If the force system is in equilibrium, R = 0.

$0 = R^2 = (\Sigma X)^2 + (\Sigma Y)^2$

$\Sigma X = 0, \Sigma Y = 0$
(If sum of the squares of two digits is zero, then each digit is zero)

Thus, the necessary and sufficient conditions of equilibrium for a system of co-planar and non-concurrent forces are:

(i) The algebraic sum of the resolved parts of the forces along any direction is equal to zero (i.e.,ΣX= 0),
(ii) The algebraic sum of the resolved parts of the forces along a directional right angles to the previous direction is equal to zero (i.e. $\Sigma Y = 0$),and
(iii) The algebraic sum of the moments of the forces about any point intheir plane is equal to zero (i.e. $\Sigma M = 0$).

TYPES OF EQUILIBRIUM

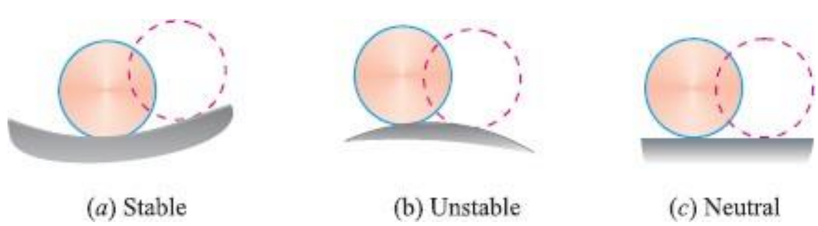


Fig 2.1

Stable equilibrium
A body is said to be in stable equilibrium, if it returns back to its original position, after it is slightly displaced from its position of rest. This happens when some additional force sets up due to displacement and brings the body back to its original position.
Unstable equilibrium
A body is said to be in an unstable equilibrium, if it does not return back to its original position, and heels farther away, after slightly displaced from its position of rest.
Neutral equilibrium
A body is said to be in a neutral equilibrium, if it occupies a new position (and remains at rest in this position) after slightly displaced from its position of rest.

Free body: A body is said to be free body if it is isolated from all other connected members

FREE BODY DIAGRAM
Free body diagram of a body is the diagram drawn by showing all the external forces and reactions on the body and by removing the contact surfaces.

Steps to be followed in drawing a free body diagram

a. Isolate the body from all other bodies.
b. Indicate the external forces on the free body. (The weight of the body should also be included. It should be applied at the centre of gravity of the body)
c. The magnitude and direction of the known external forces should be mentioned.
d. The reactions exerted by the supports on the body should be clearly indicated.
e. Clearly mark the dimensions in the free body diagram.

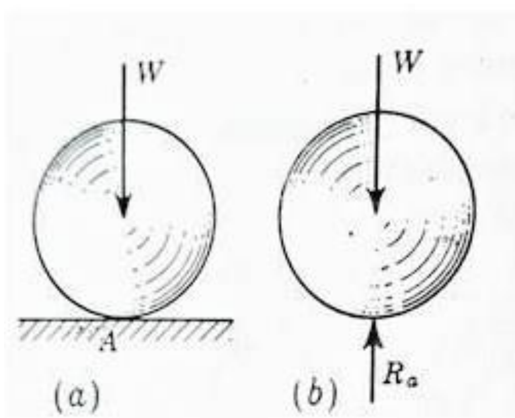


Fig 2.2

A spherical ball is rested upon a surface as shown in figure 2.2 (a). By following the necessary steps we can draw the free body diagram for this force system as shown in figure 2.2(b). Similarly fig 2.3 (b) represents free body diagram for the the force system shown in figure 2.3(a).

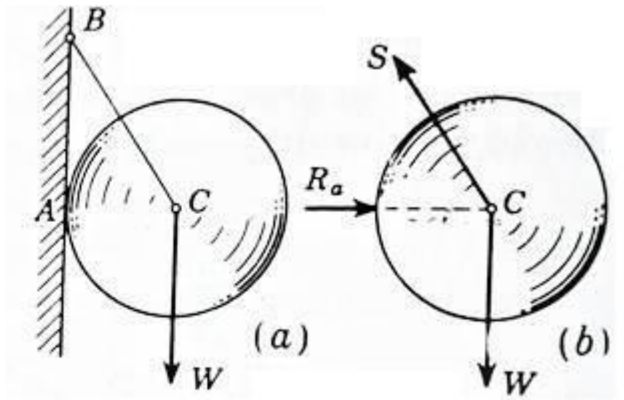


Fig 2.3

METHOD OF EQUILIBRIUM OF COPLANAR FORCES The methods of finding out equilibrium for concurrent and non-concurrent forces in a coplanar force system are:

- Analytical method
- Graphical method

ANALYTICAL METHOD:

The equilibrium of coplanar concurrent and non-concurrent forces can be studied analytically by **Lami's theorem**.

2.2 LAMI'S THEOREM

It states, "If three coplanar forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two." Mathematically,

$$\frac{P}{Sin\ \text{a}} = \frac{Q}{Sin\beta} = \frac{R}{Sin\gamma}$$

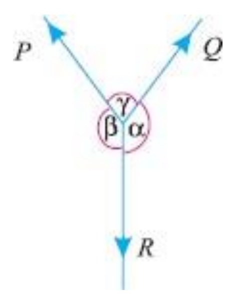


Fig 2.4

Where, *P*, *Q*, and *R*are three forces and α, β, γ are the angles as shown in Fig.
Proof:
Consider three coplanar forces *P*, *Q*, and *R* acting at a point *O*. Let the opposite angles to three forces be α , β and γ as shown in Fig.

Now let us complete the parallelogram *OACB* with *OA* and *OB* as adjacent sides as shown in the figure. We know that the resultant of two forces *P* and *Q* will be given by the diagonal *OC* both in magnitude and direction of the parallelogram *OACB*.

Since these forces are in equilibrium, therefore the resultant of the forces *P* and *Q* must be in line with *OD* and equal to *R*, but in opposite direction

From the geometry of the figure, we find

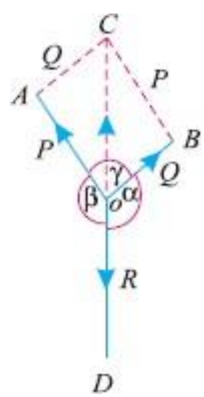


Fig 2.5

BC = *P* and *AC* = *Q*

$\therefore \angle AOC = (180° - \beta)$

and $\angle ACO = \angle BOC = (180° - \alpha)$

$\therefore \angle CAO = 180° - (\angle AOC + \angle ACO)$

$= 180° - [(180° - \beta) + (180° - \alpha)]$

$= 180° - 180° + \beta - 180° + \alpha = \alpha + \beta - 180°$

But $\alpha + \beta + \gamma = 360°$

Subtracting 180° from both sides of the above equation,

$(\alpha + \beta - 180°) + \gamma = 360° - 180° = 180°$

or $\angle CAO = 180° - \gamma$

We know that in triangle AOC

$$\frac{0A}{\sin(\angle AC0)} = \frac{AC}{\sin(\angle A0C)} = \frac{0C}{\sin(\angle CA0)}$$

$$\frac{0A}{\sin(180° - \alpha)} = \frac{AC}{\sin(180° - \beta)} = \frac{0C}{\sin(180° - \gamma)}$$

$$\frac{P}{\sin \alpha} = \frac{\mathrm{O}}{\sin \mathrm{Q}} = \frac{R}{\sin \gamma}$$

Example 2.1: An electric light fixture weighting 15 N hangs from a point C, by two strings AC and BC. The string AC is inclined at 60° to the horizontal and BC at 45° to the horizontal as shown in Fig. Using Lami's theorem, or otherwise, determine the forces in the strings AC and BC.

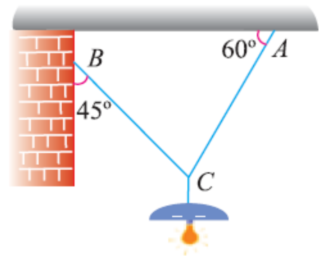


Fig 2.6

Solution.

Given:

Weight at *C* = 15 N

Let *TAC* = Force in the string *AC*, and

TBC = Force in the string *BC*.

The system of forces is shown in Fig. From the geometry of the figure, we find that angle between *TAC* and 15 N is 150° and angle between *TBC* and 15 N is 135°.

$\angle ACB = 180° - (45° + 60°) = 75°$. Applying Lami's equation at *C*,

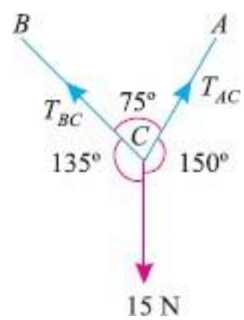


Fig 2.7

$$\frac{15}{\sin 75^\circ} = \frac{T_{AC}}{\sin 135^\circ} = \frac{T_{BC}}{\sin 150^\circ}$$

$$\frac{15}{\sin 75^\circ} = \frac{T_{AC}}{\sin 45^\circ} = \frac{T_{BC}}{\sin 30^\circ}$$

$$T_{AC} = \frac{15 \sin 45^\circ}{\sin 75^\circ} = \frac{15 \times 0.707}{0.9659} = 10.98\,\text{N}$$

$$T_{BC} = \frac{15 \sin 30^\circ}{\sin 75^\circ} = \frac{15 \times 0.5}{0.9659} = 7.76\,\text{N}$$

Example 2.2: A string ABCD, attached to fixed points A and D has two equal weights of 1000 N attached to it at B and C. The weights rest with the portions AB and CD inclined at angles as shown in Fig2.8 . Find the tensions in the portions AB, BC and CD of the string, if the inclination of the portion BC with the vertical is 120°.

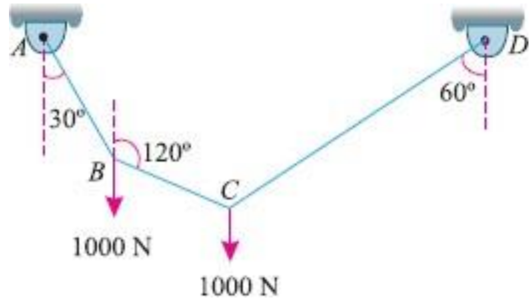


Fig 2.8

Solution: Given : Load at *B* = Load at *C* = 1000 N For the sake of convenience, let us split up the string *ABCD* into two parts. The system of forces at joints *B* and is shown in Fig.2.9 (*a*) and (*b*).

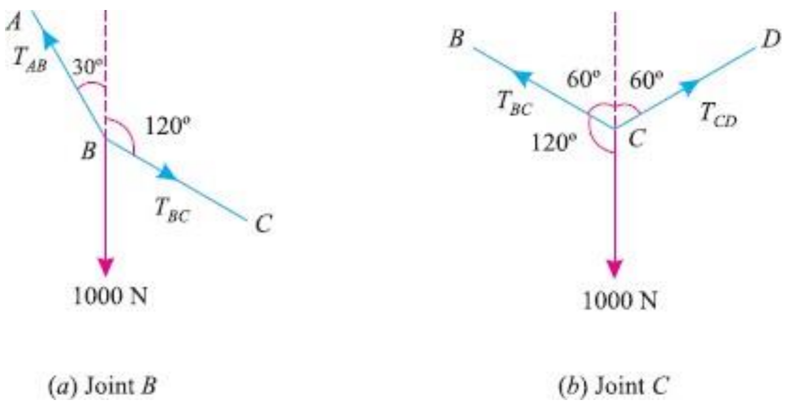


Fig 2.9

Let

TAB = Tension in the portion *AB* of the string,
TBC = Tension in the portion *BC* of the string, and
TCD = Tension in the portion *CD* of the string.

Applying Lami‟s equation at joint *B*,

$$\frac{T_{AB}}{\sin 60^\circ} = \frac{T_{BC}}{\sin 150^\circ} = \frac{1000}{\sin 150^\circ}$$

$$\frac{T_{AB}}{\sin 60^\circ} = \frac{T_{BC}}{\sin 30^\circ} = \frac{1000}{\sin 30^\circ} \qquad ...[\because \sin(180^\circ - \theta) = \sin\theta]$$

$$T_{AB} = \frac{1000 \sin 60^\circ}{\sin 30^\circ} = \frac{1000 \times 0.866}{0.5} = 1732 \text{ N} \quad \textbf{Ans.}$$

$$T_{BC} = \frac{1000 \sin 30^\circ}{\sin 30^\circ} = 1000 \text{ N} \quad \textbf{Ans.}$$

Again applying Lami‟s equation at joint *C*,

$$\frac{T_{BC}}{\sin 120^\circ} = \frac{T_{CD}}{\sin 120^\circ} = \frac{1000}{\sin 120^\circ}$$

$$T_{CD} = \frac{1000 \sin 120^\circ}{\sin 120^\circ} = 1000\,\text{N} \quad \textbf{Ans.}$$

Example 2.3. A light string ABCDE whose extremity A is fixed, has weights W1 and W2 attached to it at B and C. It passes round a small smooth peg at D carrying a weight of 300 N at the free end E as shown in Fig 2.10 below. If in the equilibrium position, BC is horizontal and AB and CD make 150° and 120° with BC, find (i) Tensions in the portion AB, BC and CD of the string and (ii) Magnitudes of $W_1$and W_2.

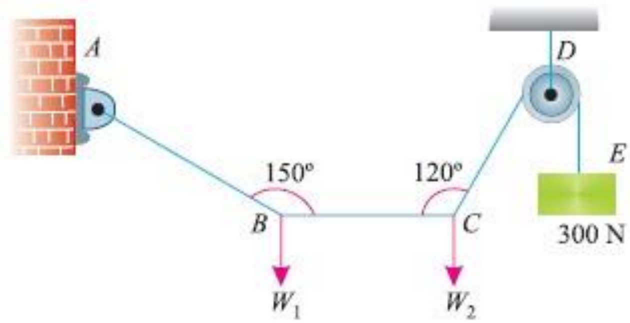


Fig 2.10

Solution: Given: Weight at *E* = 300 N For the sake of convenience, let us split up the string *ABCD* into two parts. The system of forces at joints *B* and *C* is shown in Fig (*a*) and (*b*).

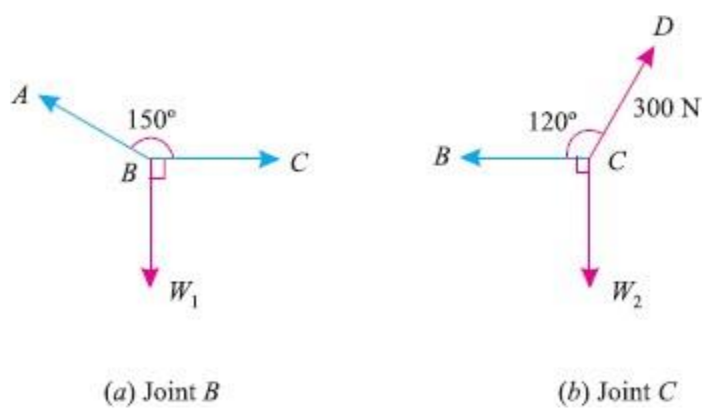


Fig 2.11

(i) Tensions is the portion AB, BC and CD of the string

Let *TAB* = Tension in the portion *AB*, and

TBC = Tension in the portion *BC*,

We know that tension in the portion *CD* of the string.

TCD = *TDE* = 300 N **Ans.**

Applying Lami‟s equation at C,

$$\frac{T_{BC}}{\sin 150^\circ} = \frac{W_2}{\sin 120^\circ} = \frac{300}{\sin 90^\circ}$$

$$\frac{T_{BC}}{\sin 30^\circ} = \frac{W_2}{\sin 60^\circ} = \frac{300}{1} \quad \text{...[}\because \sin(180^\circ - \theta) = \sin\theta\text{]}$$

$$T_{BC} = 300 \sin 30^\circ = 300 \times 0.5 = 150 \text{ N} \quad \textbf{Ans.}$$

$$W_2 = 300 \sin 60^\circ = 300 \times 0.866 = 259.8 \text{ N}$$

Again applying Lami"s equation at *B*,

$$\frac{T_{AB}}{\sin 90^\circ} = \frac{W_1}{\sin 150^\circ} = \frac{T_{BC}}{\sin 120^\circ}$$

$$\frac{T_{AB}}{1} = \frac{W_1}{\sin 30^\circ} = \frac{150}{\sin 60^\circ} \quad \text{...[}\because \sin(180^\circ - \theta) = \sin\theta\text{]}$$

$$T_{AB} = \frac{150}{\sin 60^\circ} = \frac{150}{0.866} = 173.2 \text{ N} \quad \textbf{Ans.}$$

$$W_1 = \frac{150 \sin 30^\circ}{\sin 60^\circ} = \frac{150 \times 0.5}{0.866} = 86.6 \text{ N}$$

(ii) *Magnitudes of W*1 *and W*2

From the above calculations, we find that the magnitudes of *W*1 and *W*2 are 86.6 N and 259.8 N respectively.

Example 2.4 Two cylinders P and Q rest in a channel as shown in Fig 2.12. The cylinder P has diameter of 100 mm and weighs 200 N, whereas the cylinder Q has diameter of 180 mm and weighs 500 N. If the bottom width of the box is 180 mm, with one side vertical and the other inclined at 60°, determine the pressures at all the four points of contact.

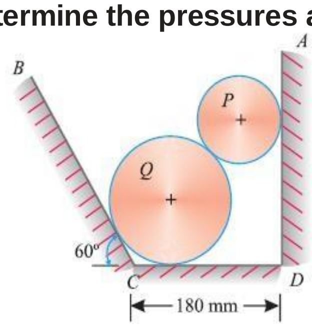


Fig 2.12

Soln.:Given : Diameter of cylinder *P* = 100 mm ; Weight of cylinder *P* = 200 N ;
Diameter of cylinder *Q* = 180 mm ; Weight of cylinder *Q* = 500 N and width of channel = 180 mm.

First of all, consider the equilibrium of the cylinder *P*

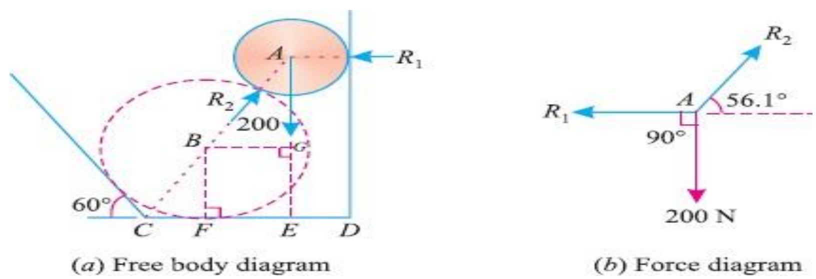


Fig 2.13

From the geometry of the figure, we find that
ED= Radius of cylinder P = 100/2 =50 mm
BF = Rdius of cylinder Q = 180/2 = 90mm
<*BCF* = 60°
$\therefore$ *CF* =*BF* cot 60° = 90×0.577 = 52 mm
FE = BG = 180 - (52+50) = 78 mm
AB = 50+90 = 140 mm
Cos ∠ABG = BG/AB = 78/140 = 0.5571
∠ABG = 56.1°
Applying Lami"s theorem at A,

$$\frac{R_1}{\sin(90°+56.1°)} = \frac{R_2}{\sin 90°} = \frac{200}{\sin(180°-56.1°)}$$

$$\frac{R_1}{\cos 56.1°} = \frac{R_2}{1} = \frac{200}{\sin 56.1°}$$

$$\therefore R_1 = \frac{200\cos 56.1°}{\sin 56.1°} = \frac{200\times 0.5571}{0.830} = 134.2\text{N}$$

$$\text{And } R_2 = \frac{200}{\sin 56.1°} = \frac{200}{0.8300} = 240.8\text{N}$$

Example 2.5Three cylinders weighting 100 N each and of 80 mm diameter are placed in a channel of 180 mm width as shown in Fig. Determine the pressure exerted by (i) the cylinder A on B at the point of contact (ii) the cylinder B on the base and (iii) the cylinder B on the wall.

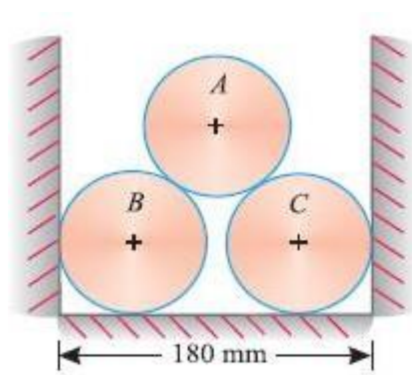


Fig 2.14

Solution. Given: Weight of each cylinder = 100 N; Dia. of each cylinder = 80 mm and width of channel = 180 mm

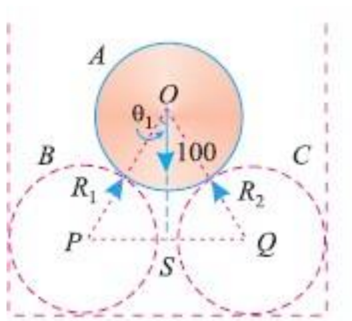
(i) Pressure exerted by the cylinder A on the cylinder B

Let R_1 = Pressure exerted by the cylinder *A*on *B*. It is also equal to pressure exerted by the cylinder *A*on *B*.

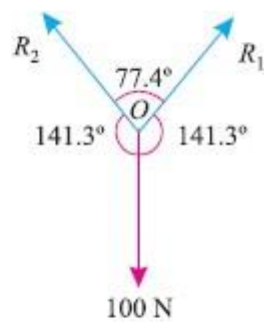
First of all, consider the equilibrium of the cylinder *A*. It is in equilibrium under the action of the following forces, which must pass through the centre of the cylinder as shown in Fig 2.15 (*a*).

1. Weight of the cylinder 100 N acting downwards.
2. Reaction *R*1 of the cylinder *B* on the cylinder *A*.
3. Reaction *R*2 of the cylinder *C* on the cylinder *A*.

Now join the centres *O*, *P* and *Q* of the three cylinders. Bisect *PQ* at *S* and join *OS* as shown in Fig 2.15 (*b*).



(*a*) Free body diagram



(*b*) Force diagram

Fig 2.15

From the geometry of the triangle OPS, we find that

OP = 40 + 40 = 80 mm

and PS = 90 – 40 = 50 mm

$$\sin \angle POS = \frac{PS}{OP} = \frac{50}{80} = 0.625$$

$$\angle POS = 38.7°$$

Since the triangle OSQ is similar to the triangle OPS, therefore ∠SOQ is also equal to 38.7°. Thus the angle between R1 and R2 is 2 × 38.7° = 77.4°. And angle between R1 and OS (also between R2 and OS). = 180° – 38.7° = 141.3°

The system of forces at O is shown in Fig (b).

Applying Lami"s equation at O,

$$\frac{R_1}{\sin 141.3°} = \frac{R_2}{\sin 141.3°} = \frac{100}{\sin 77.4°}$$

$$\frac{R_1}{\sin 38.7°} = \frac{R_2}{\sin 38.7°} = \frac{100}{\sin 77.4°} \qquad \text{...[} \because \sin(180° - \theta) = \sin\theta \text{]}$$

$$R_1 = \frac{100 \times \sin 38.7°}{\sin 77.4°} = \frac{100 \times 0.6252}{0.9759} = 64.0 \text{ N} \quad \textbf{Ans.}$$

$$R_2 = R_1 = 64.0 \text{ N} \quad \textbf{Ans.}$$

(ii) Pressure exerted by the cylinder B on the base

Let R_3 = Pressure exerted by the cylinder B on the wall, and

R_4 = Pressure exerted by the cylinder B on the base.

Now consider the equilibrium of the cylinder *B*. It is in equilibrium under the action of the following forces, which must pass through the centre of the cylinder as shown in Fig 2.16 (*a*).

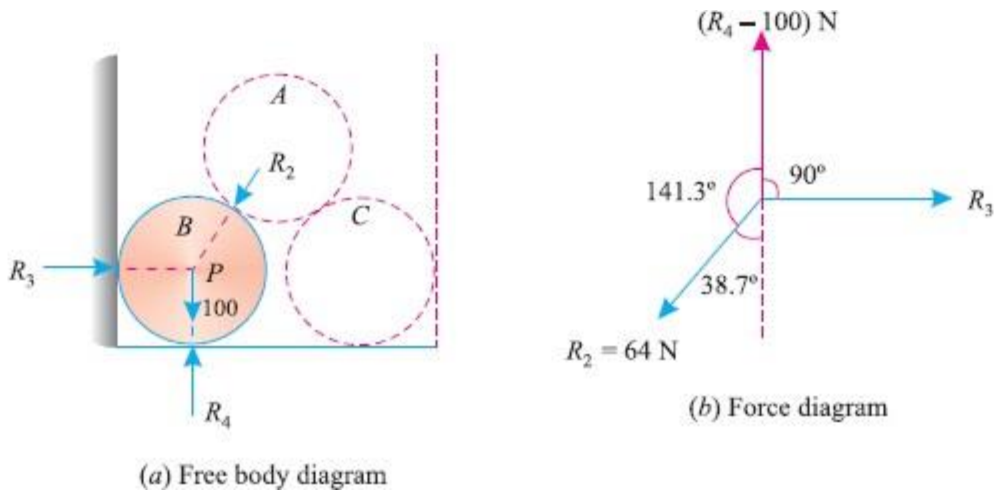


Fig 2.16

1. Weight of the cylinder 100 N acting downwards.
2. Reaction R2 equal to 64.0 N of the cylinder A on the cylinder B.
3. Reaction R3 of the cylinder B on the vertical side of the channel.
4. Reaction R4 of the cylinder B on the base of the channel.

A little consideration will show that weight of the cylinder B is acting downwards and the reaction R4 is acting upwards. Moreover, their lines of action also coincide with each other. Therefore net downward force will be equal to (R_4 – 100) N. The system of forces is shown in Fig 2.16 (b). Applying Lami"s equation at P,

$$\frac{64}{\sin 90^\circ} = \frac{R_3}{\sin (180^\circ - 38.7^\circ)} = \frac{(R_4 - 100)}{\sin (90^\circ + 38.7^\circ)}$$

$$\frac{64}{1} = \frac{R_3}{\sin 38.7^\circ} = \frac{R_4 - 100}{\cos 38.7^\circ}$$

$$R_4 - 100 = 64 \cos 38.7^\circ = 64 \times 0.7804 = 50 \text{ N}$$

R_4 = 50 + 100 = 150 N **Ans.**

(iii) Pressure exerted by the cylinder B on the wall. From the above Lami"s equation, we also find that

R_3 = 64 sin 38.7° = 64 × 0.6252 = 40 N **Ans.**

Note. Since the cylinders B and C are symmetrically placed, therefore pressures exerted by the cylinder C on the wall as well as channel will be the same as those exerted by the cylinder *B*.

GRAPHICAL METHOD FOR THE EQUILIBRIUM OF COPLANAR FORCES

The equilibrium of coplanar forces may also be studied, graphically, by drawing the vector diagram. This may also be done by studying the

1. Converse of the Law of Triangle of Forces.
2. Converse of the Law of Polygon of Forces.

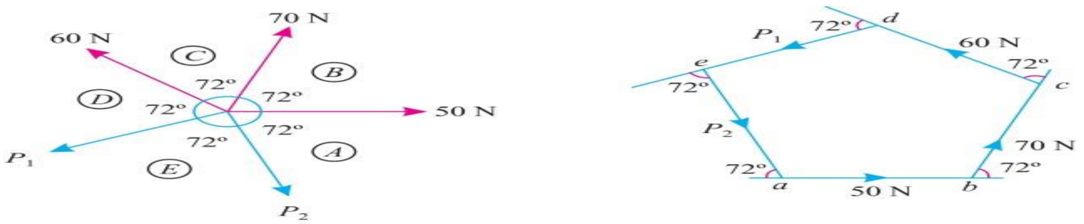
CONVERSE OF THE LAW OF TRIANGLE OF FORCES

If three forces acting at a point be represented in magnitude and direction by the three sides a triangle, taken in order, the forces shall be in equilibrium.

CONVERSE OF THE LAW OF POLYGON OF FORCES

If any number of forces acting at a point be represented in magnitude and direction by the sides of a closed polygon, taken in order, the forces shall be in equilibrium.

Example 2.6 Five strings are tied at a point and are pulled in all directions, equally spaced from one another. If the magnitude of the pulls on three consecutive strings is 50 N, 70 N and 60 N respectively, find graphically the magnitude of the pulls on two other strings.
Solution. Given : Pulls = 50 N ; 70 N and 60 N and angle between the forces = 360/5=72
Let *P*1 and *P*2 = Pulls in the two strings.
First of all, let us draw the space diagram for the given system of forces and name them according to Bow‟s notations as shown in Fig(a)



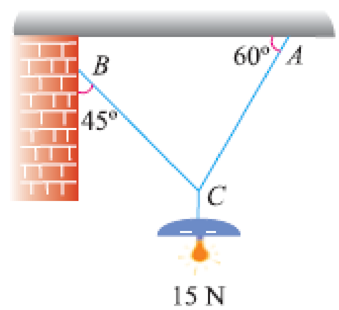
(*a*) Space diagram (*b*) Vector diagram

Fig 2.17

Now draw the vector diagram for the given forces as shown in Fig 2.17 (*b*) and as discussed below :
1. Select some suitable point *a* and draw a horizontal line *ab* equal to 50 N to some suitable scale representing the force *AB*.
2. Through *b* draw a line *bc* equal to 70 N to the scale and parallel to *BC*.
3. Similarly through *c*, draw *cd* equal to 60 N to the scale and parallel to *CD*.
4. Through *d* draw a line parallel to the force *P*1 of the space diagram.
5. Similarly through *a* draw a line parallel to the force *P*2 meeting the first line at *e*, thus closing the polygon *abcde*, which means that the point is in equilibrium.
6. By measurement, we find that the forces *P*1 = 57.5 N and *P*2 = 72.5 N respectively. **Ans**

EXERCISE

1. Enunciate any two principles of equilibrium.
2. State and prove Lami‟s Theorem.
3. Show that if three coplaner forces, acting at a point be in equilibrium, then, each force is proportional to the sine of the angle between the other two.
4. What are different methods of studying the equilibrium of coplaner forces ? Describe any one of them.
5. How would you find out the equilibrium of non-coplaner forces ?
6. Explain the conditions of equilibrium.
7. Discuss the various types of equilibrium.
8. Two equal heavy spheres of 50 mm radius are in equilibrium within a smooth cup of 150 mm radius. Show that the reaction between the cups of one sphere is double than that between the two spheres.
9. A smooth circular cylinder of radius 1.5 meter is lying in a triangular groove, one side of which makes 15° angle and the other 40° angle with the horizontal. Find the reactions atthe surfaces of contact, if there is no friction and the cylinder weights 100 N. **Ans. 78.5, 31.6**
10. An electric light fixture weighing 15 N hangs from a point C, by two strings AC and BC. The string AC is inclined at 60° to the horizontal and BC at 45° to the horizontal as shown in Fig.**Ans. 1N, 7.8N**



Truss/ Frame: A pin jointed frame is a structure made of slender (cross-sectional dimensions quite small compared to length) members pin connected at ends and capable of taking load at joints.

Such frames are used as roof trusses to support sloping roofs and as bridge trusses to support deck.

Plane frame: A frame in which all members lie in a single plane is called plane frame. They are designed to resist the forces acting in the plane of frame. Roof trusses and bridge trusses are the example of plane frames.

Space frame: If all the members of frame do not lie in a single plane, they are called as space frame. Tripod, transmission towers are the examples of space frames.

Perfect frame: A pin jointed frame which has got just sufficient number of members to resist the loads without undergoing appreciable deformation in shape is called a perfect frame. Triangular frame is the simplest perfect frame and it has 03 joints and 03 members.

It may be observed that to increase one joint in a perfect frame, two more members are required. Hence, the following expression may be written as the relationship between number of joint j, and the number of members m in a perfect frame.

$$m = 2j - 3$$

(a) When LHS = RHS, Perfect frame.
(b) When LHS<RHS, Deficient frame.
(c) When LHS>RHS, Redundant frame.

Assumptions

The following assumptions are made in the analysis of pin jointed trusses:

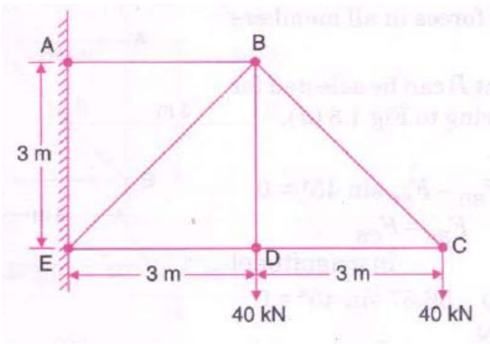
1. The ends of the members are pin jointed (hinged).
2. The loads act only at the joints.
3. Self weight of the members is negligible.

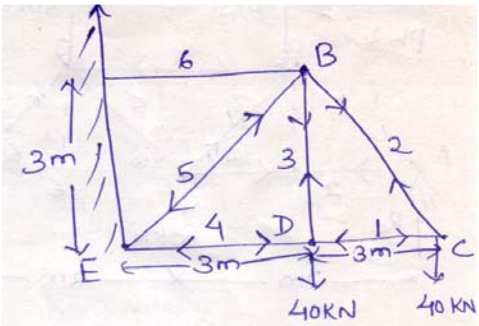
Methods of analysis

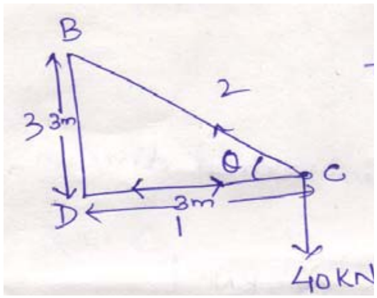
1. Method of joint
2. Method of section

Problems on method of joints

Problem 1: Find the forces in all the members of the truss shown in figure.



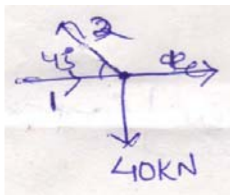




$\tan\theta = 1$

$\Rightarrow \theta = 45^\circ$

Joint C



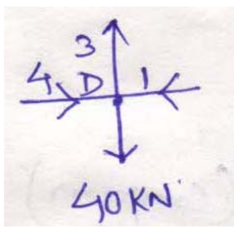
$S_1 = S_2 \cos 45$

$\Rightarrow S_1 = 40KN$ (Compression)

$S_2 \sin 45 = 40$

$\Rightarrow S_2 = 56.56KN$ (Tension)

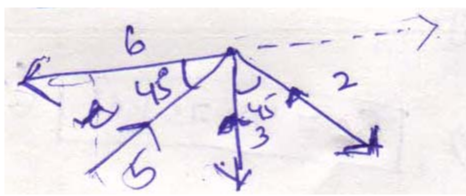
Joint D



$S_3 = 40KN$ (Tension)

$S_1 = S_4 = 40KN$ (Compression)

Joint B



Resolving vertically,

$\sum V = 0$

$S_5 \sin 45 = S_3 + S_2 \sin 45$

$\Rightarrow S_5 = 113.137KN$ (Compression)

Resolving horizontally,

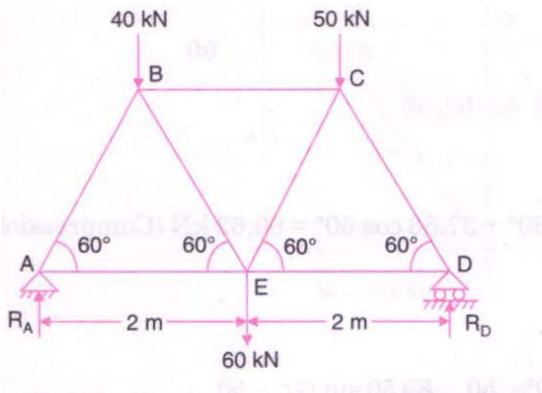
$\sum H = 0$

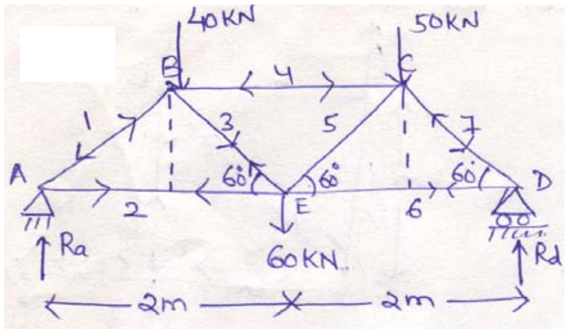
$S_6 = S_5 \cos 45 + S_2 \cos 45$

$\Rightarrow S_6 = 113.137 \cos 45 + 56.56 \cos 45$

$\Rightarrow S_6 = 120KN$ (Tension)

Problem 2: Determine the forces in all the members of the truss shown in figure and indicate the magnitude and nature of the forces on the diagram of the truss. All inclined members are at 60° to horizontal and length of each member is 2m.





Taking moment at point A,

$\sum M_A = 0$

$R_d \times 4 = 40 \times 1 + 60 \times 2 + 50 \times 3$

$\Rightarrow R_d = 77.5KN$

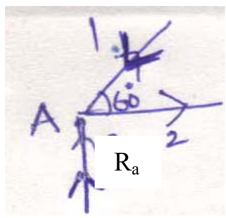
Now resolving all the forces in vertical direction,

$\sum V = 0$

$R_a + R_d = 40 + 60 + 50$

$\Rightarrow R_a = 72.5KN$

<u>Joint A</u>



$\sum V = 0$

$\Rightarrow R_a = S_1 \sin 60$

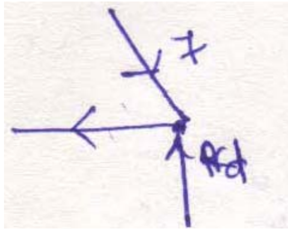
$\Rightarrow S_1 = 83.72KN$ (Compression)

$\sum H = 0$

$\Rightarrow S_2 = S_1 \cos 60$

$\Rightarrow S_1 = 41.86KN$ (Tension)

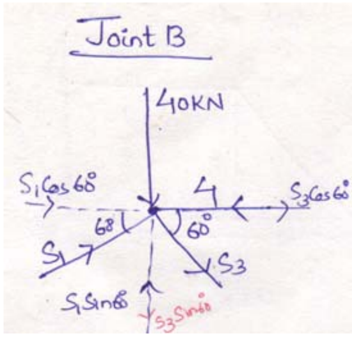
Joint D



$\sum V = 0$
$S_7 \sin 60 = 77.5$
$\Rightarrow S_7 = 89.5KN$ (Compression)

$\sum H = 0$
$S_6 = S_7 \cos 60$
$\Rightarrow S_6 = 44.75KN$ (Tension)

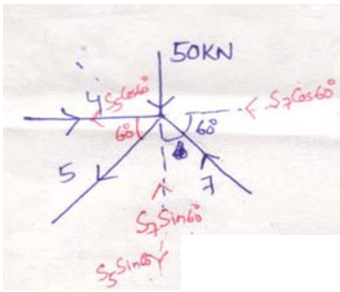
Joint B



$\sum V = 0$
$S_1 \sin 60 = S_3 \cos 60 + 40$
$\Rightarrow S_3 = 37.532KN$ (Tension)

$\sum H = 0$
$S_4 = S_1 \cos 60 + S_3 \cos 60$
$\Rightarrow S_4 = 37.532 \cos 60 + 83.72 \cos 60$
$\Rightarrow S_4 = 60.626KN$ (Compression)

Joint C



$\sum V = 0$
$S_5 \sin 60 + 50 = S_7 \sin 60$
$\Rightarrow S_5 = 31.76KN$ (Tension)

Plane Truss (Method of Section)

In case of analysing a plane truss, using method of section, after determining the support reactions a section line is drawn passing through not more than three members in which forces are unknown, such that the entire frame is cut into two separate parts.

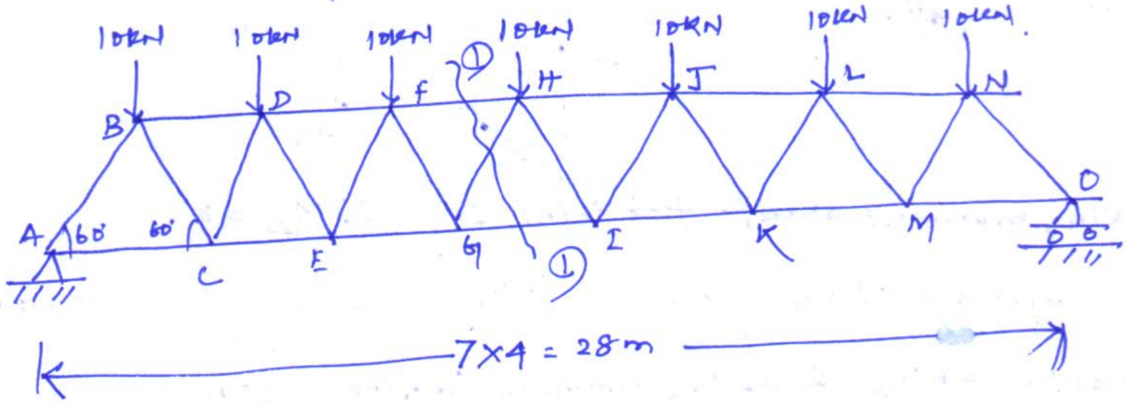
Each part should be in equilibrium under the action of loads, reactions and the forces in the members.

Method of section is preferred for the following cases:

(i) analysis of large truss in which forces in only few members are required

(ii) If method of joint fails to start or proceed with analysis for not getting a joint with only two unknown forces.

Example 1.

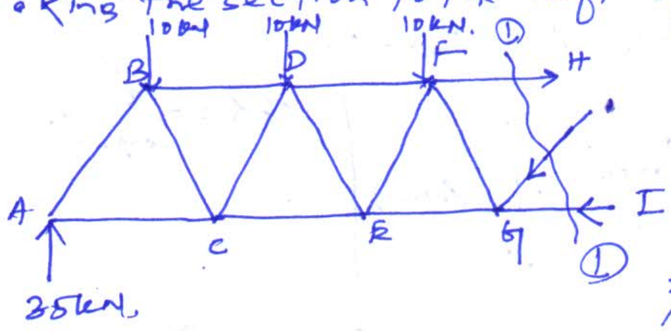


Determine the forces in the members FH, HG, and GI in the truss

Due to symmetry $R_A = R_O = \frac{1}{2} \times$ total downward load

$= \frac{1}{2} \times 70 = \boxed{35 KN.}$

Taking the section to the left of the cut.



Taking moment about G

$\Sigma M_G = 0.$

$F_{FH} \times 4\sin 60 + 35 \times 12$

$= 10\times 2 + 10\times 6 + 10\times 10$

$\Rightarrow F_{FH} = \dfrac{(20+60+100) - 420}{4\sin 60}$

$= -69.28$ kN.

Negative sign indicates that direction should have opposite i.e it is compressive in nature.

Now Resolving all the forces vertically $\Sigma Y = 0$

$$10 + 10 + 10 + F_{GH} \sin 60 = 35$$

$$\Rightarrow F_{GH} = \frac{35 - 30}{\sin 60^\circ}$$

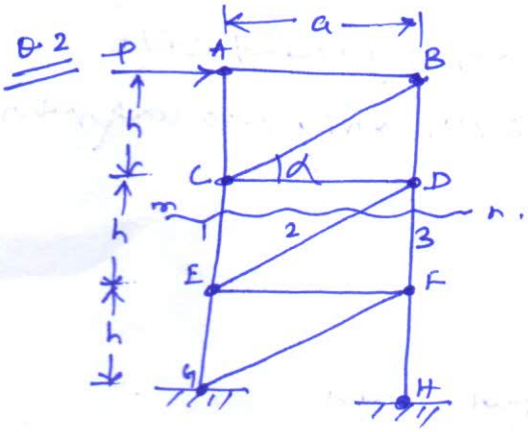
$$\Rightarrow \boxed{F_{GH} = 5.78 \text{ kN.}} \text{ (compressive)}$$

Resolving all the forces horizontally $\Sigma X = 0$.

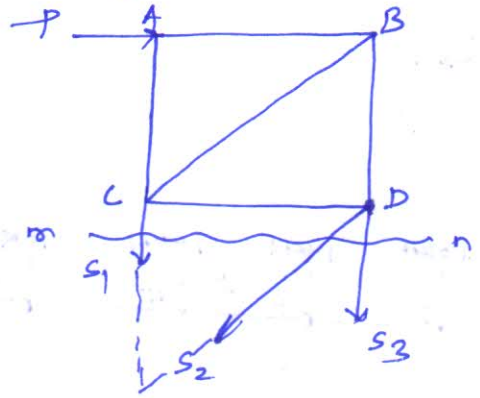
$$F_{FH} + F_{GH} \cos 60 = F_{GI}$$

$$\Rightarrow F_{GI} = 69.28 + 5.78 \cos 60^\circ = \boxed{72.17 \text{ kN.}} \text{ (tension)}$$

Q.2



Using method of sections determine the axial forces in bars 1, 2 and 3.



Taking moment about ~~poi~~ joint D $\Sigma M_D = 0$.

$$S_1 \times a = P \times h \Rightarrow \boxed{S_1 = \frac{Ph}{a}} \quad \text{— (1)} \quad \text{(tension)}$$

Similarly taking E as the moment centre $\Sigma M_E = 0$

$$S_3 \times a + P \times 2h = 0$$

$$\Rightarrow \boxed{S_3 = \frac{-2Ph}{a}}$$

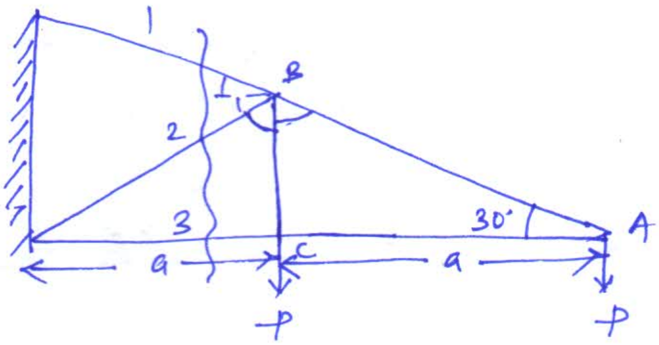
(–ve sign indicates direction of force will be opposite and it will be compressive in nature)

Resolving all the forces horizontally. $\Sigma X = 0$.

$$S_2 \cos\alpha = P$$

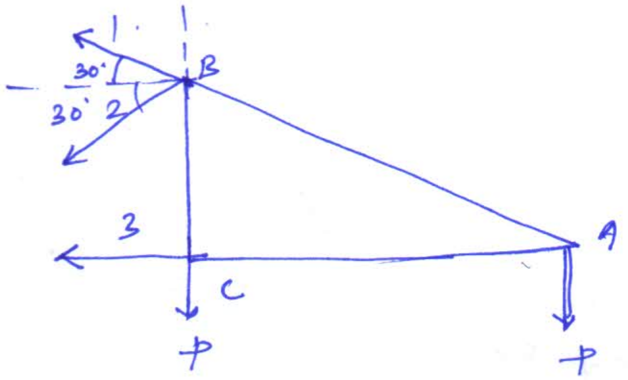
$$\Rightarrow S_2 = \frac{P}{\cos\alpha} = \boxed{\frac{P\sqrt{a^2+h^2}}{a}} \text{ (Ans)}$$

$$\cos\alpha = \frac{a}{\sqrt{a^2+h^2}}$$



$\frac{BC}{AC} = \tan 30°$

$\Rightarrow BC = a \tan 30 = \boxed{0.578a}$



$\Sigma M_B = 0.$

$S_3 \times 0.578a + P \times a = 0$

$\Rightarrow S_3 = \frac{-Pa}{0.578a} = -1.73P$

(-ve sign indicates direction is opposite and it is compressive in nature)

Resolving vertically, $\Sigma Y = 0$

$S_1 \sin 30 = 2P + S_2 \sin 30$

$\Rightarrow S_1 = \frac{2P + S_2/2}{\sin 30} = (4P + S_2) \quad — (2)$

Now resolving horizontally, $\Sigma X = 0.$

$S_1 \cos 30 + S_2 \cos 30 = 1.73P$

$\Rightarrow (4P + S_2) \times \frac{\sqrt{3}}{2} + S_2 \frac{\sqrt{3}}{2} = 1.73P$

$\Rightarrow 2\sqrt{3}P + \frac{\sqrt{3}}{2} S_2 + \frac{\sqrt{3}}{2} S_2 = 1.73P$

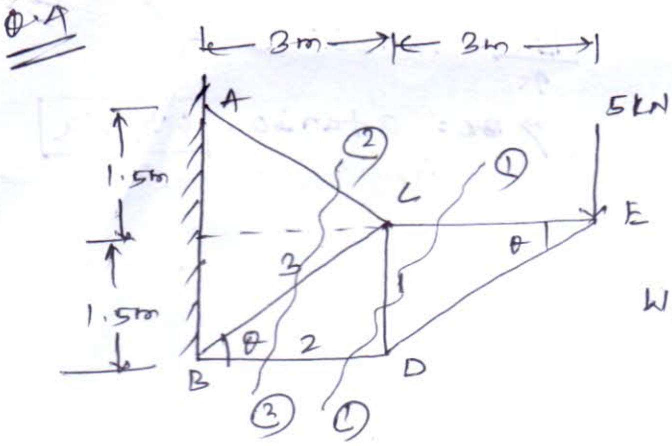
$\Rightarrow \frac{\sqrt{3}}{2} S_2 = 1.73P - 2\sqrt{3}P$

$= -1.73P$

$\Rightarrow S_2 = \frac{-1.73P}{\sqrt{3}} = \boxed{-P}$ (-ve sign indicates the direction is opposite and it is compressive)

Now $S_1 = 4P - P = \boxed{3P}$ (tension)

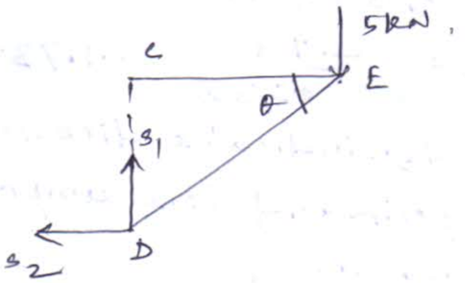
Q.4



Using method of sections, find axial forces in each bar 1, 2 and 3 of the plane truss.

We have $\tan\theta = \left(\frac{1.5}{3}\right) \Rightarrow \boxed{\theta = 26.56^\circ}$

__Considering section 1-1__



Resolving vertically, $\Sigma Y = 0$

$\boxed{S_1 = 5kN}$ (tension)

Now taking moment about C

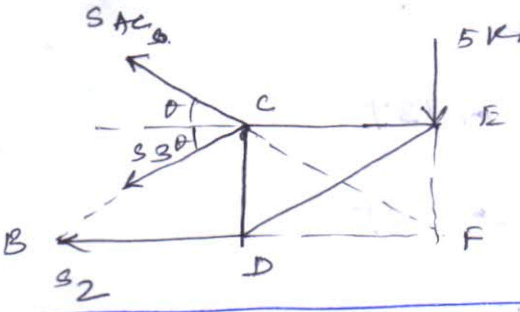
$S_2 \times 1.5 + 5 \times 3 = 0$

$\Rightarrow S_2 = -10kN$

-ve sign indicates direction should have been opposite

$\boxed{S_2 = 10kN}$ (compression)

__Considering section 2-2__

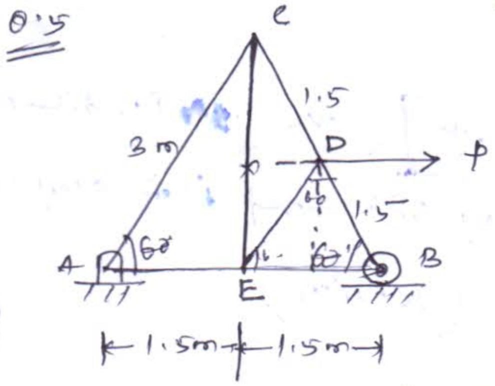


Taking moment about F

$\Sigma M_F = 0$

$\Rightarrow \boxed{S_3 = 0}$

Q.5



__Assignment__

Using method of joint and method of section find the axial force in the bar x.

__Method of Joint__

Considering the whole structure and taking moment about A $\Sigma M_A = 0$.

$R_B \times 3 = P \times 1.5 \sin 60$

$\Rightarrow \boxed{R_B = \frac{\sqrt{3}}{4} P.}$

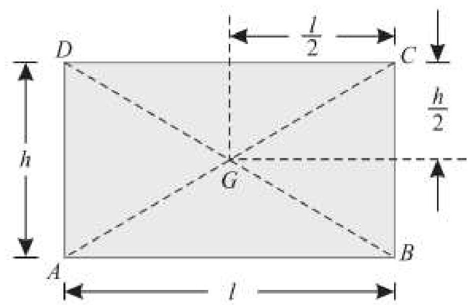
4. CENTROID AND MOMENT OF INERTIA

CENTRE OF GRAVITY: The point, through which the whole weight of the body acts, irrespective of its position, is known as centre of gravity (briefly written as C.G.). It may be noted that everybody has one and only one centre of gravity.

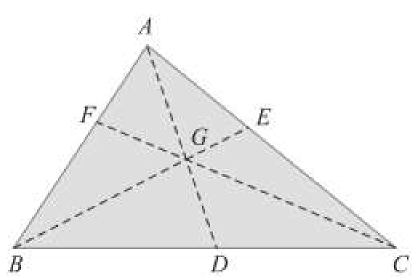
CENTROID: The plane figures (like triangle, quadrilateral, circle etc.) have only areas, but no mass. The centre of area of such figures is known as centroid. The method of finding out the centroid of a figure is the same as that of finding out the centre of gravity of a body.

CENTRE OF GRAVITY BY GEOMETRICAL CONSIDERATIONS:
The centre of gravity of simple figures may be found out from the geometry of the figure as given below.

1. The centre of gravity of uniform rod is at its middle point.
2. The centre of gravity of a rectangle (or a parallelogram) is at the point, where its diagonals meet each other. It is also a middle point of the length as well as the breadth of the rectangle as shown in Fig.



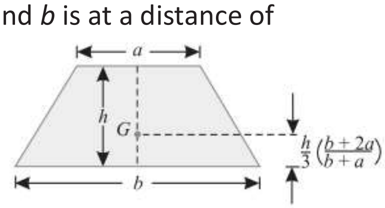
3. The centre of gravity of a triangle is at the point, where the three medians (a median is a line connecting the vertex and middle point of the opposite side) of the triangle meet as shown in Fig.



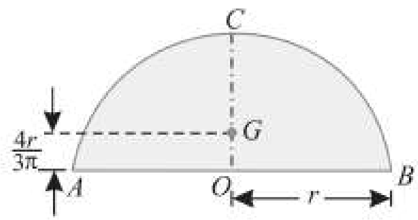
4. The centre of gravity of a trapezium with parallel sides *a* and *b* is at a distance of

$$\frac{h}{3} \times \left(\frac{b+2a}{b+a}\right)$$

measured form the side *b* as shown in Fig.



5. The centre of gravity of a semicircle is at a distance of 4r/3 π from its base measured along the vertical radius as shown in Fig.



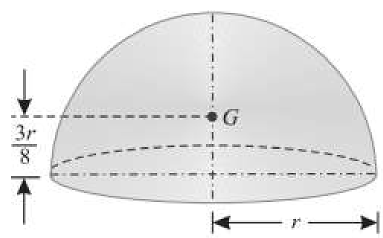
6. The centre of gravity of a circular sector making semi-vertical angle α is at a distance of

$$\frac{2r}{3}\frac{\sin\alpha}{\alpha}$$

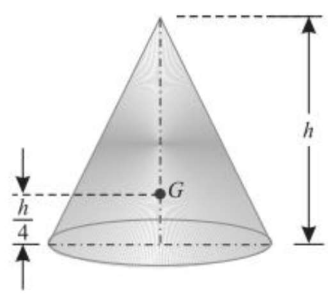
7. The centre of gravity of a cube is at a distance of *l*/ 2 from every face (where *l* is the length of each side).

8. The centre of gravity of a sphere is at a distance of *d* /2 from every point (where *d* is the diameter of the sphere).

9. The centre of gravity of a hemisphere is at a distance of 3*r*/ 8 from its base, measured along the vertical radius as shown in Fig.



10. The centre of gravity of right circular solid cone is at a distance of *h* /4 from its base, measured along the vertical axis as shown in Fig.



AXIS OF REFERENCE:

The centre of gravity of a body is always calculated with reference to some assumed axis known as axis of reference (or sometimes with reference to some point of reference). The axis of reference, of plane figures, is generally taken as the lowest line of the figure for calculating $\bar{y}$ and the left line of the figure for calculating $\bar{x}$.

CENTRE OF GRAVITY OF PLANE FIGURES:

Let $\overline{x}$ and $\overline{y}$ be the co-ordinates of the centre of gravity with respect to some axis of reference, then

$$\overline{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots\ldots}{a_1 + a_2 + a_3}$$

and

$$\overline{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \ldots\ldots}{a_1 + a_2 + a_3 + \ldots}$$

where a_1, a_2, a_3........ etc., are the areas into which the whole figure is divided x_1, x_2, x_3 ..... etc., are the respective co-ordinates of the areas a_1, a_2, a_3....... on *X-X* axis with respect to same axis of reference.

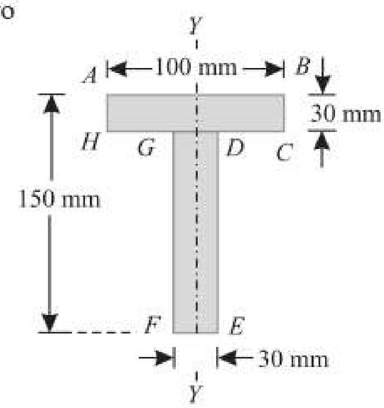
y_1, y_2, y_3....... etc., are the respective co-ordinates of the areas a_1, a_2, a_3....... on *Y-Y* axis with respect to same axis of the reference.

CENTRE OF GRAVITY OF SYMMETRICAL SECTIONS:

Sometimes, the given section, whose centre of gravity is required to be found out, is symmetrical about *X-X* axis or *Y-Y* axis. In such cases, the procedure for calculating the centre of gravity of the body is very much simplified; as we have only to calculate either $\overline{x}$ or $\overline{y}$. This is due to the reason that the centre of gravity of the body will lie on the axis of symmetry.

EXAMPLE: Find the centre of gravity of a 100 mm × 150 mm × 30 mm T-section.

Solution. As the section is symmetrical about *Y-Y* axis, bisecting the web, therefore its centre of gravity will lie on this axis. Split up the section into two rectangles *ABCH* and *DEFG* as shown in Fig 6.10.



Let bottom of the web *FE* be the axis of reference.

(*i*) *Rectangle ABCH*

$$a_1 = 100 \times 30 = 3000 \text{ mm}^2$$

and

$$y_1 = \left(150 - \frac{30}{2}\right) = 135 \text{ mm}$$

(*ii*) *Rectangle DEFG*

$$a_2 = 120 \times 30 = 3600 \text{ mm}^2$$

and

$$y_2 = \frac{120}{2} = 60 \text{ mm}$$

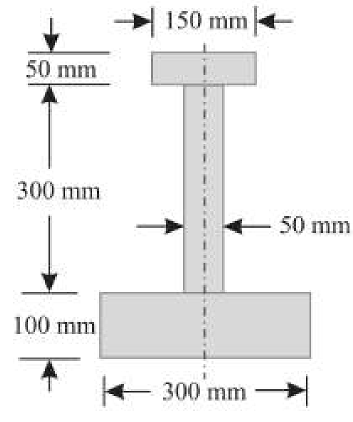
We know that distance between centre of gravity of the section and bottom of the flange *FE*,

$$\overline{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 135) + (3600 \times 60)}{3000 + 3600} \text{ mm}$$

$$= 94.1 \text{ mm} \quad \textbf{Ans.}$$

EXAMPLE: An I-section has the following dimensions in mm units:
Bottom flange = 300 × 100
Top flange = 150 × 50
Web = 300 × 50
Determine mathematically the position of centre of gravity of the section.

Solution. As the section is symmetrical about Y-Y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Now split up the section into three rectangles as shown in Fig.



Let bottom of the bottom flange be the axis of reference.

(*i*) *Bottom flange*

$$a_1 = 300 \times 100 = 30\,000 \text{ mm}^2$$

and $$y_1 = \frac{100}{2} = 50 \text{ mm}$$

(*ii*) *Web*

$$a_2 = 300 \times 50 = 15\,000 \text{ mm}^2$$

and $$y_2 = 100 + \frac{300}{2} = 250 \text{ mm}$$

(*iii*) *Top flange*

$$a_3 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_3 = 100 + 300 + \frac{50}{2} = 425 \text{ mm}$$

We know that distance between centre of gravity of the section and bottom of the flange,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(30\,000 \times 50) + (15\,000 \times 250) + (7500 \times 425)}{30\,000 + 15\,000 + 7500} = 160.7 \text{ mm} \quad \textbf{Ans.}$$

CENTRE OF GRAVITY OF UNSYMMETRICAL SECTIONS:

Sometimes, the given section, whose centre of gravity is required to be found out, is not symmetrical either about X-X axis or Y-Y axis. In such cases, we have to find out both the values of $\bar{x}$ and $\bar{y}$

EXAMPLE: Find the centroid of an unequal angle section 100 mm × 80 mm × 20 mm.

Solution. As the section is not symmetrical about any axis, therefore we have to find out the values of $\bar{x}$ and $\bar{y}$ for the angle section. Split up the section into two rectangles as shown in Fig.

Let left face of the vertical section and bottom face of the horizontal section be axes of reference.

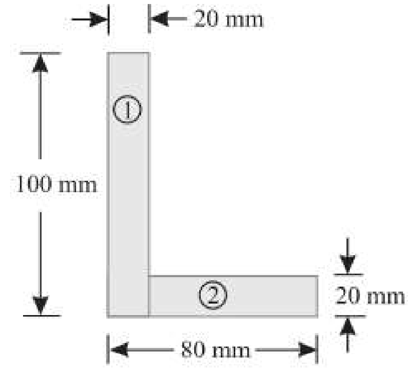
Let left face of the vertical section and bottom face of the horizontal section be axes of reference.

(*i*) *Rectangle* 1

$$a_1 = 100 \times 20 = 2000 \text{ mm}^2$$

$$x_1 = \frac{20}{2} = 10 \text{ mm}$$

and $$y_1 = \frac{100}{2} = 50 \text{ mm}$$



(*ii*) *Rectangle* 2

$$a_2 = (80 - 20) \times 20 = 1200 \text{ mm}^2$$

$$x_2 = 20 + \frac{60}{2} = 50 \text{ mm}$$

and $$y_2 = \frac{20}{2} = 10 \text{ mm}$$

We know that distance between centre of gravity of the section and left face,

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(2000 \times 10) + (1200 \times 50)}{2000 + 1200} = 25 \text{ mm} \quad \textbf{Ans.}$$

Similarly, distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{} = \frac{(2000 \times 50) + (1200 \times 10)}{} = 35 \text{ mm} \quad \textbf{Ans.}$$

EXAMPLE: A body consists of a right circular solid cone of height 40 mm and radius 30 mm placed on a solid hemisphere of radius 30 mm of the same material. Find the position of centre of gravity of the body.

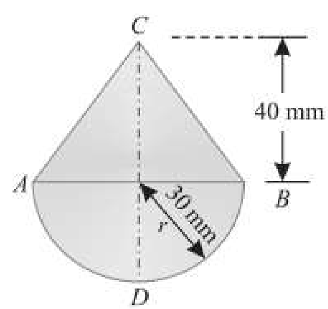
Solution. As the body is symmetrical about *Y-Y* axis, therefore its centre of gravity will lie on this axis as shown in Fig. Let bottom of the hemisphere (*D*) be the point of reference.

(*i*) *Hemisphere*

$$v_1 = \frac{2\pi}{3} \times r^3 = \frac{2\pi}{3}(30)^3 \text{ mm}^3$$

$$= 18\,000\,\pi \text{ mm}^3$$

and $$y_1 = \frac{5r}{8} = \frac{5 \times 30}{8} = 18.75 \text{ mm}$$



(*ii*) *Right circular cone*

$$v_2 = \frac{\pi}{3} \times r^2 \times h = \frac{\pi}{3} \times (30)^2 \times 40 \text{ mm}^3$$

$$= 12\,000\,\pi \text{ mm}^3$$

and $$y_2 = 30 + \frac{40}{4} = 40 \text{ mm}$$

We know that distance between centre of gravity of the body and bottom of hemisphere *D*,

$$\bar{y} = \frac{v_1 y_1 + v_2 y_2}{v_1 + v_2} = \frac{(18\,000\pi \times 18.75) + (12\,000\pi \times 40)}{18\,000\pi + 12\,000\pi} \text{ mm}$$

$$= 27.3 \text{ mm} \quad \textbf{Ans.}$$

MOMENT OF INERTIA: The moment of a force (*P*) about a point, is the product of the force and perpendicular distance (*x*) between the point and the line of action of the force (*i.e*. *P*.*x*). This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance (*x*) between the point and the line of action of the force *i.e*. *P*.*x* (*x*) = *Px*2, then this quantity is called moment of the moment of a force or second moment of force or moment of inertia (briefly written as M.I.).

MOMENT OF INERTIA OF A PLANE AREA:

Consider a plane area, whose moment of inertia is required to be found out. Split up the whole area into a number of small elements.

Let $a_1, a_2, a_3, \ldots =$ Areas of small elements, and

$r_1, r_2, r_3, \ldots =$ Corresponding distances of the elements from the line about which the moment of inertia is required to be found out.

Now the moment of inertia of the area,

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \ldots$$
$$= \Sigma\, a\, r^2$$

UNITS OF MOMENT OF INERTIA:

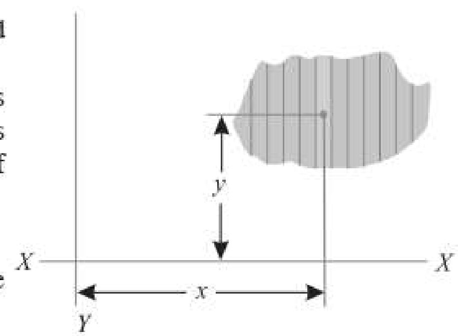
As a matter of fact the units of moment of inertia of a plane area depend upon the units of the area and the length. e.g.

1. If area is in m^2 and the length is also in m, the moment of inertia is expressed in m^4
2. If area in mm^2 and the length is also in mm, then moment of inertia is expressed in mm^4.

MOMENT OF INERTIA BY INTEGRATION:

The moment of inertia of an area may also be found out by the method of integration as discussed below:

Consider a plane figure, whose moment of inertia is required to be found out about *X-X* axis and *Y-Y* axis as shown in Fig Let us divide the whole area into a no. of strips. Consider one of these strips.



. Moment of inertia by integration.

Let $dA =$ Area of the strip

$x =$ Distance of the centre of gravity of the strip on *X-X* axis and

$y =$ Distance of the centre of gravity of the strip on *Y-Y* axis.

We know that the moment of inertia of the strip about *Y-Y* axis

$$= dA \,.\, x^2$$

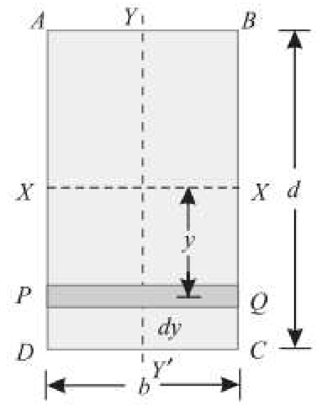
Now the moment of inertia of the whole area may be found out by integrating above equation. *i.e.*,

$$I_{YY} = \Sigma\, dA \,.\, x^2$$

Similarly $I_{XX} = \Sigma\, dA \,.\, y^2$

MOMENT OF INERTIA OF A RECTANGULAR SECTION:

Consider a rectangular section *ABCD* as shown in Fig. whose moment of inertia is required to be found out.



Rectangular section.

Let b = Width of the section and

d = Depth of the section.

Now consider a strip *PQ* of thickness dy parallel to *X-X* axis and at a distance y from it as shown in the figure

∴ Area of the strip

$= b.dy$

We know that moment of inertia of the strip about *X-X* axis,

$= \text{Area} \times y^2 = (b.\,dy)\,y^2 = b.\,y^2.\,dy$

Now *moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina *i.e.* from $-\frac{d}{2}$ to $+\frac{d}{2}$,

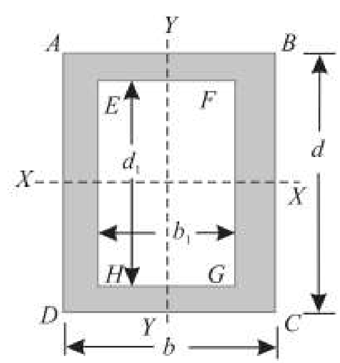
$$I_{xx} = \int_{-\frac{d}{2}}^{+\frac{d}{2}} b.y^2.dy = b\int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2\,.\,dy$$

$$= b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = b\left[\frac{(d/2)^3}{3} - \frac{(-d/2)^3}{3}\right] = \frac{bd^3}{12}$$

Similarly, $I_{YY} = \frac{db^3}{12}$

MOMENT OF INERTIA OF A HOLLOW RECTANGULAR SECTION:

Consider a hollow rectangular section, in which *ABCD* is the main section and *EFGH* is the cut out section as shown in Fig



Hollow rectangular section.

Let b = Breadth of the outer rectangle,

d = Depth of the outer rectangle and

b_1, d_1 = Corresponding values for the cut out rectangle.

We know that the moment of inertia, of the outer rectangle *ABCD* about *X-X* axis

$$= \frac{bd^3}{12} \quad \ldots(i)$$

and moment of inertia of the cut out rectangle *EFGH* about *X-X* axis

$$= \frac{b_1 d_1^3}{12} \quad \ldots(ii)$$

∴ M.I. of the hollow rectangular section about *X-X* axis,

I_{XX} = M.I. of rectangle *ABCD* – M.I. of rectangle *EFGH*

$$= \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$$

Similarly, $I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$

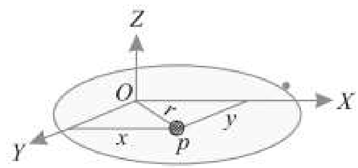
THEOREM OF PERPENDICULAR AXIS:

It states, *If I_{XX} and I_{YY} be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia I_{ZZ} about the axis Z-Z, perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:*

$$I_{ZZ} = I_{XX} + I_{YY}$$

Proof :

Consider a small lamina (P) of area da having co-ordinates as x and y along OX and OY two mutually perpendicular axes on a plane section as shown in Fig.



Theorem of perpendicular axis.

Now consider a plane OZ perpendicular to OX and OY. Let (r) be the distance of the lamina (P) from Z-Z axis such that $OP = r$.

From the geometry of the figure, we find that

$$r^2 = x^2 + y^2$$

We know that the moment of inertia of the lamina P about X-X axis,

$$I_{XX} = da.\, y^2 \qquad \ldots[\because I = \text{Area} \times (\text{Distance})^2]$$

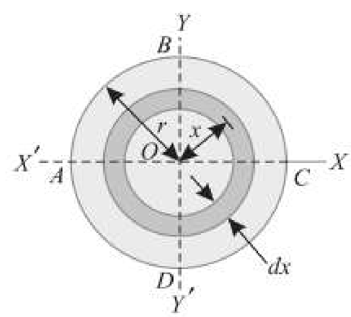
Similarly, $$I_{YY} = da.\, x^2$$

and $$I_{ZZ} = da.\, r^2 = da\,(x^2 + y^2) \qquad \ldots(\because r^2 = x^2 + y^2)$$

$$= da.\, x^2 + da.\, y^2 = I_{YY} + I_{XX}$$

MOMENT OF INERTIA OF A CIRCULAR SECTION:

Consider a circle $ABCD$ of radius (r) with centre O and X-X' and Y-Y' be two axes of reference through O as shown in Fig.



Circular section.

Now consider an elementary ring of radius x and thickness dx. Therefore area of the ring,

$$da = 2\,\pi\, x.\, dx$$

and moment of inertia of ring, about X-X axis or Y-Y axis

$$= \text{Area} \times (\text{Distance})^2$$
$$= 2\,\pi\, x.\, dx \times x^2$$
$$= 2\,\pi\, x^3.\, dx$$

Now moment of inertia of the whole section, about the central axis, can be found out by integrating the above equation for the whole radius of the circle *i.e.*, from 0 to r.

$$\therefore \qquad I_{ZZ} = \int_0^r 2\pi x^3\, .\, dx = 2\pi \int_0^r x^3\, .\, dx$$

$$I_{ZZ} = 2\pi \left[\frac{x^4}{4}\right]_0^r = \frac{\pi}{2}(r)^4 = \frac{\pi}{32}(d)^4 \qquad \ldots\left(\text{substituting } r = \frac{d}{2}\right)$$

We know from the Theorem of Perpendicular Axis that

$$I_{XX} + I_{YY} = I_{ZZ}$$

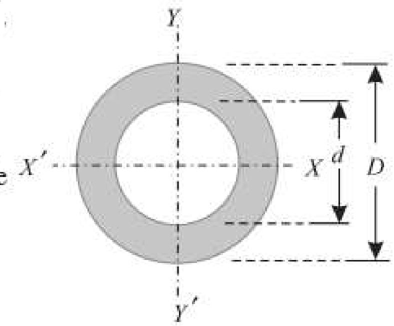
$$* I_{XX} = I_{YY} = \frac{I_{ZZ}}{2} = \frac{1}{2} \times \frac{\pi}{32}(d)^4 = \frac{\pi}{64}(d)^4$$

MOMENT OF INERTIA OF A HOLLOW CIRCULAR SECTION:

Consider a hollow circular section as shown in Fig. whose moment of inertia is required to be found out.

Let D = Diameter of the main circle, and

d = Diameter of the cut out circle.



Hollow circular section.

We know that the moment of inertia of the main circle about X-X axis

$$= \frac{\pi}{64}(D)^4$$

and moment of inertia of the cut-out circle about X-X axis

$$= \frac{\pi}{64}(d)^4$$

∴ Moment of inertia of the hollow circular section about X-X axis,

I_{XX} = Moment of inertia of main circle – Moment of inertia of cut out circle,

$$= \frac{\pi}{64}(D)^4 - \frac{\pi}{64}(d)^4 = \frac{\pi}{64}(D^4 - d^4)$$

Similarly, $$I_{YY} = \frac{\pi}{64}(D^4 - d^4)$$

THEOREM OF PARALLEL AXIS:

It states, *If the moment of inertia of a plane area about an axis through its centre of gravity is denoted by I_G, then moment of inertia of the area about any other axis* ***AB****, parallel to the first, and at a distance h from the centre of gravity is given by:*

$$I_{AB} = I_G + ah^2$$

where I_{AB} = Moment of inertia of the area about an axis AB,

I_G = Moment of Inertia of the area about its centre of gravity

a = Area of the section, and

h = Distance between centre of gravity of the section and axis AB.

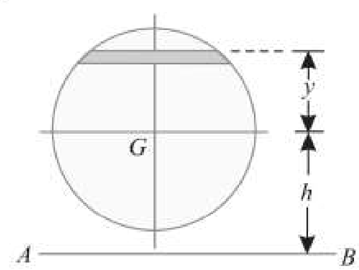
Proof

Consider a strip of a circle, whose moment of inertia is required to be found out about a line AB as shown in Fig.

Let δa = Area of the strip

y = Distance of the strip from the centre of gravity the section and

h = Distance between centre of gravity of the section and the axis AB.



Theorem of parallel axis.

We know that moment of inertia of the whole section about an axis passing through the centre of gravity of the section

$$= \delta a . y^2$$

and moment of inertia of the whole section about an axis passing through its centre of gravity,

$$I_G = \Sigma\, \delta a.\, y^2$$

∴ Moment of inertia of the section about the axis AB,

$$I_{AB} = \Sigma\, \delta a\,(h+y)^2 = \Sigma\, \delta a\,(h^2 + y^2 + 2\,h\,y)$$
$$= (\Sigma\, h^2.\, \delta a) + (\Sigma\, y^2.\, \delta a) + (\Sigma\, 2\,h\,y\,.\, \delta a)$$
$$= a\,h^2 + I_G + 0$$

It may be noted that $\Sigma\, h^2\,.\,\delta a = a\,h^2$ and $\Sigma\, y^2\,.\,\delta a = I_G$ [as per equation (i) above] and $\Sigma\, \delta a.y$ is the algebraic sum of moments of all the areas, about an axis through centre of gravity of the section and is equal to $a.\overline{y}$, where $\overline{y}$ is the distance between the section and the axis passing through the centre of gravity, which obviously is zero.

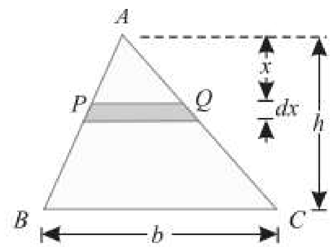
MOMENT OF INERTIA OF A TRIANGULAR SECTION:

Consider a triangular section ABC whose moment of inertia is required to be found out.

Let b = Base of the triangular section and

h = Height of the triangular section.

Now consider a small strip PQ of thickness dx at a distance of x from the vertex A as shown in Fig. From the geometry of the figure, we find that the two triangles APQ and ABC are similar. Therefore



. Triangular section.

$$\frac{PQ}{BC} = \frac{x}{h} \quad \text{or} \quad PQ = \frac{BC.x}{h} = \frac{bx}{h} \qquad (\because BC = \text{base} = b)$$

We know that area of the strip PQ

$$= \frac{bx}{h}\,.\,dx$$

and moment of inertia of the strip about the base BC

$$= \text{Area} \times (\text{Distance})^2 = \frac{bx}{h}\,dx\,(h-x)^2 = \frac{bx}{h}\,(h-x)^2\,dx$$

Now moment of inertia of the whole triangular section may be found out by integrating the above equation for the whole height of the triangle $i.e.$, from 0 to h.

$$I_{BC} = \int_0^h \frac{b\,x}{h}\,(h-x)^2\,dx$$

$$= \frac{b}{h} \int_0^h x\,(h^2 + x^2 - 2hx)\,dx$$

$$= \frac{b}{h} \int_0^h (xh^2 + x^3 - 2hx^2)\,dx$$

$$= \frac{b}{h} \left[\frac{x^2 h^2}{2} + \frac{x^4}{4} - \frac{2hx^3}{3} \right]_0^h = \frac{bh^3}{12}$$

We know that distance between centre of gravity of the triangular section and base *BC*,

$$d = \frac{h}{3}$$

$\therefore$ Moment of inertia of the triangular section about an axis through its centre of gravity and parallel to *X-X* axis,

$$I_G = I_{BC} - ad^2 \qquad \ldots(\because I_{XX} = I_G + a\,h^2)$$

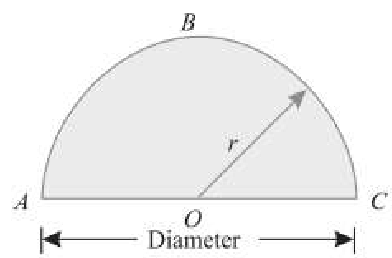
$$= \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

MOMENT OF INERTIA OF A SEMICIRCULAR SECTION:

Consider a semicircular section *ABC* whose moment of inertia is required to be found out as shown in Fig. ....

Let r = Radius of the semicircle.

We know that moment of inertia of the semicircular section about the base *AC* is equal to half the moment of inertia of the circular section about *AC*. Therefore moment of inertia of the semicircular section *ABC* about the base *AC*,



Semicircular section *ABC*.

$$I_{AC} = \frac{1}{2} \times \frac{\pi}{64} \times (d)^4 = 0.393\,r^4$$

We also know that area of semicircular section,

$$a = \frac{1}{2} \times \pi r^2 \;\; \frac{\pi r^2}{2}$$

and distance between centre of gravity of the section and the base *AC*,

$$h = \frac{4r}{3\pi}$$

$\therefore$ Moment of inertia of the section through its centre of gravity and parallel to *x-x* axis,

$$I_G = I_{AC} - ah^2 = \left[\frac{\pi}{8} \times (r)^4\right] - \left[\frac{\pi r^2}{2}\left(\frac{4r}{3\pi}\right)^2\right]$$

$$= \left[\frac{\pi}{8} \times (r)^4\right] - \left[\frac{8}{9\pi} \times (r)^4\right] = 0.11\,r^4$$

Note. The moment of inertia about *y-y* axis will be the same as that about the base *AC* *i.e.*, $0.393\,r^4$.

MOMENT OF INERTIA OF A COMPOSITE SECTION:

The moment of inertia of a composite section may be found out by the following steps :

1. First of all, split up the given section into plane areas (*i.e.*, rectangular, triangular, circular etc., and find the centre of gravity of the section).
2. Find the moments of inertia of these areas about their respective centres of gravity.
3. Now transfer these moment of inertia about the required axis (*AB*) by the Theorem of Parallel Axis, *i.e.*,

$$I_{AB} = I_G + ah^2$$

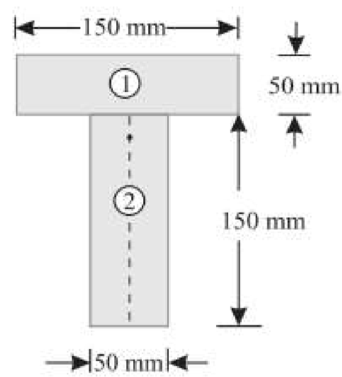
where I_G = Moment of inertia of a section about its centre of gravity and parallel to the axis.
a = Area of the section,
h = Distance between the required axis and centre of gravity of the section.

4. The moments of inertia of the given section may now be obtained by the algebraic sum of the moment of inertia about the required axis.

EXAMPLE: Find the moment of inertia of a T-section with flange as 150 mm × 50 mm and web as 150 mm × 50 mm about X-X and Y-Y axes through the centre of gravity of the section.

Solution. The given *T*-section is shown in Fig.

First of all, let us find out centre of gravity of the section. As the section is symmetrical about *Y-Y* axis, therefore its centre of gravity will lie on this axis. Split up the whole section into two rectangles *viz*., 1 and 2 as shown in figure. Let bottom of the web be the axis of reference.



(*i*) *Rectangle* (1)

$$a_1 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_1 = 150 + \frac{50}{2} = 175 \text{ mm}$$

(*ii*) *Rectangle* (2)

$$a_2 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_2 = \frac{150}{2} = 75 \text{ mm}$$

We know that distance between centre of gravity of the section and bottom of the web,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(7500 \times 175) + (7500 \times 75)}{7500 + 7500} = 125 \text{ mm}$$

Moment of inertia about X-X axis

We also know that M.I. of rectangle (1) about an axis through its centre of gravity and parallel to *X-X* axis.

$$I_{G1} = \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and *X-X* axis,

$$h_1 = 175 - 125 = 50 \text{ mm}$$

∴ Moment of inertia of rectangle (1) about *X-X* axis

$$I_{G1} + a_1 h_1^2 = (1.5625 \times 10^6) + [7500 \times (50)^2] = 20.3125 \times 10^6 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to *X-X* axis,

$$I_{G2} = \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and *X-X* axis,

$$h_2 = 125 - 75 = 50 \text{ mm}$$

∴ Moment of inertia of rectangle (2) about *X-X* axis

$$= I_{G2} + a_2 h_2^2 = (14.0625 \times 10^6) + [7500 \times (50)^2] = 32.8125 \times 10^6 \text{ mm}^4$$

Now moment of inertia of the whole section about *X-X* axis,

$$I_{XX} = (20.3125 \times 10^6) + (32.8125 \times 10^6) = 53.125 \times 10^6 \text{ mm}^4$$ **Ans.**

Moment of inertia about Y-Y axis

We know that M.I. of rectangle (1) about *Y-Y* axis

$$= \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and moment of inertia of rectangle (2) about *Y-Y* axis,

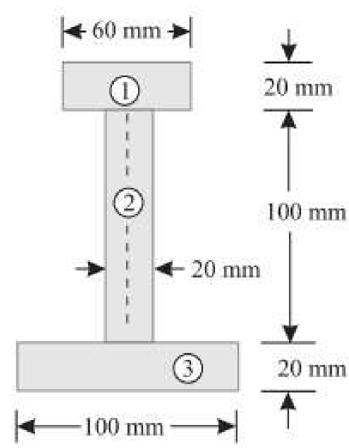
$$= \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

Now moment of inertia of the whole section about *Y-Y* axis,

$$I_{YY} = (14.0625 \times 10^6) + (1.5625 \times 10^6) = 15.625 \times 10^6 \text{ mm}^4$$ **Ans.**

EXAMPLE: An I-section is made up of three rectangles as shown in Fig. Find the moment of inertia of the section about the horizontal axis passing through the centre of gravity of the section.

Solution. First of all, let us find out centre of gravity of the section. As the section is symmetrical about *Y-Y* axis, therefore its centre of gravity will lie on this axis. Split up the whole section into three rectangles 1, 2 and 3 as shown in Fig. , Let bottom face of the bottom flange be the axis of reference.



(*i*) *Rectangle* 1

$$a_1 = 60 \times 20 = 1200 \text{ mm}$$

and $$y_1 = 20 + 100 + \frac{20}{2} = 130 \text{ mm}$$

(*ii*) *Rectangle* 2

$$a_2 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_2 = 20 + \frac{100}{2} = 70 \text{ mm}$$

(*iii*) *Rectangle* 3

$$a_3 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_3 = \frac{20}{2} = 10 \text{ mm}$$

We know that the distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(1200 \times 130) + (2000 \times 70) + (2000 \times 10)}{1200 + 2000 + 2000} \text{ mm}$$

$$= 60.8 \text{ mm}$$

We know that moment of inertia of rectangle (1) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G1} = \frac{60 \times (20)^3}{12} = 40 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and X-X axis,

$$h_1 = 130 - 60.8 = 69.2 \text{ mm}$$

∴ Moment of inertia of rectangle (1) about X-X axis,

$$= I_{G1} + a_1 h_1^2 = (40 \times 10^3) + [1200 \times (69.2)^2] = 5786 \times 10^3 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G2} = \frac{20 \times (100)^3}{12} = 1666.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and X-X axis,

$$h_2 = 70 - 60.8 = 9.2 \text{ mm}$$

∴ Moment of inertia of rectangle (2) about X-X axis,

$$= I_{G2} + a_2 h_2^2 = (1666.7 \times 10^3) + [2000 \times (9.2)^2] = 1836 \times 10^3 \text{ mm}^4$$

Now moment of inertia of rectangle (3) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G3} = \frac{100 \times (20)^3}{12} = 66.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (3) and X-X axis,

$$h_3 = 60.8 - 10 = 50.8 \text{ mm}$$

∴ Moment of inertia of rectangle (3) about X-X axis,

$$= I_{G3} + a_3 h_3^2 = (66.7 \times 10^3) + [2000 \times (50.8)^2] = 5228 \times 10^3 \text{ mm}^4$$

Now moment of inertia of the whole section about X-X axis,

$$I_{XX} = (5786 \times 10^3) + (1836 \times 10^3) + (5228 \times 10^3) = 12\,850 \times 10^3 \text{ mm}^4$$ **Ans.**

Beams :- UNIT - 2

Classification :-

Based on Equilibrium condition :-

1. Statically determinate
2. Statically Indeterminate.

Statically determinate beam are those beam which only required equilibrium Equation.

Statically Indeterminate beam are those beam which are not only solve by Equilibrium Equation it is required solve by other method.

When the no. of unknowns are equal to the number of Equilibrium Equation, it is called statically determinate Beam.

When the no. of unknowns are ~~is~~ not equal to the no. of equilibrium Equation it is called statically Indeterminate Beam.

* Degree of Indetermination (D.O.I)

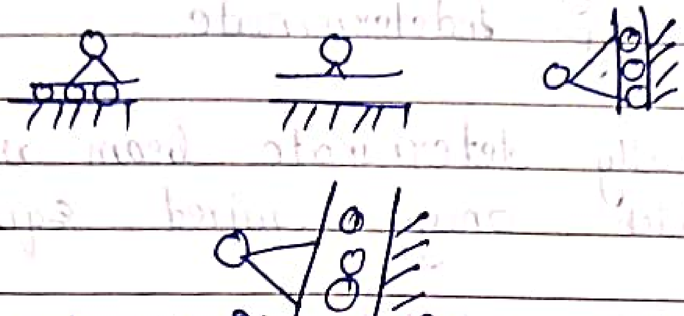
D.O.I = No. of unknowns - No. of equilibrium eqn - No. of internal hinge.

↑ It is only for Statically ~~Inte~~ Indeterminate Beams

→ Types of Support :-

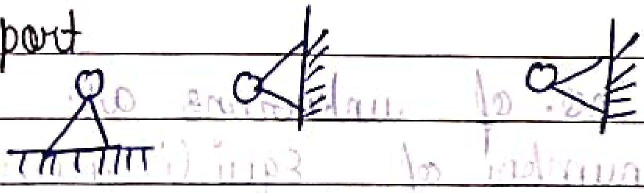
There are three types of Support.

i). Roller Support :-



इस Roller Support में केवल Horizontal Reaction आएगा या केवल vertical reaction & moment (M) = 0

ii). Hinge Support



vertical & horizontal both reaction are present & Moment = 0.

iii). Fixed Support

Horizontal Reaction, vertical Reaction and Moment all are present in fixed support.

Equation of Equilibrium

i) $\Sigma V = 0$; V = all vertical forces

ii) $\Sigma H = 0$; H = all horizontal forces

iii) $\Sigma M = 0$; M = Both Clockwise & anticlockwise Moment

Bending Moment and Shear force

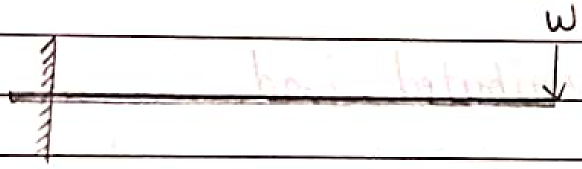
*. **Beam :-** A structural member which is acted upon by a system of external loads at right angles to its axis is known as beam.

Types of Beams and loadings :-

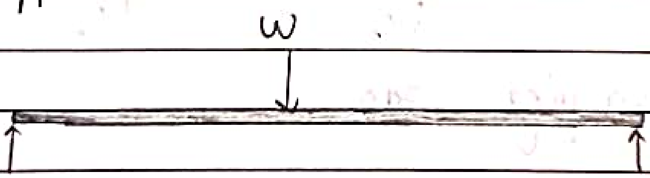
The types of beams are classified as under:

Based on Type of Support.

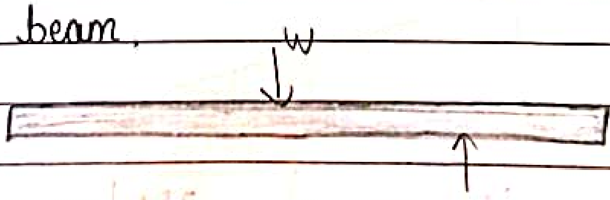
1. Cantilever beam.



2. Simply Supported beam.

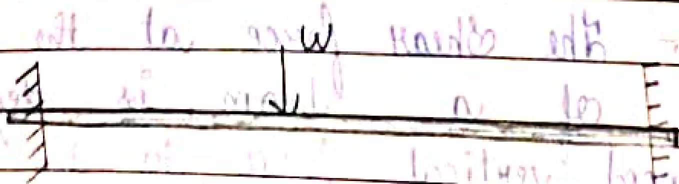


3. Overhanging beam.



} Statically Determinate Beams

4. Fixed beam.



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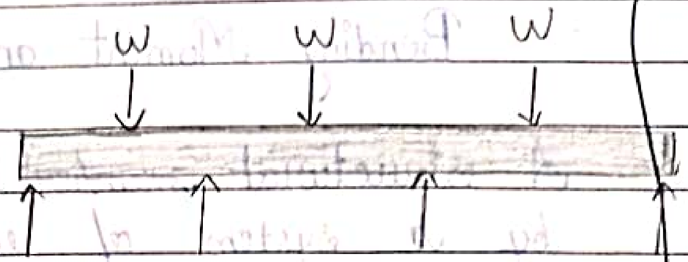
6. Propped Cantileve Beam.

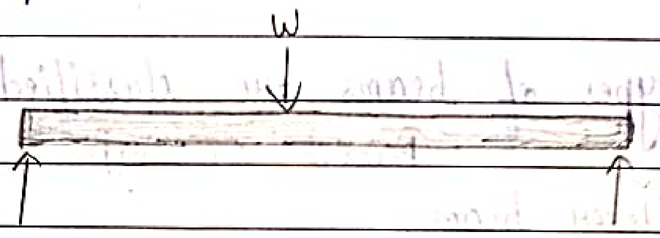
5. Continous beam



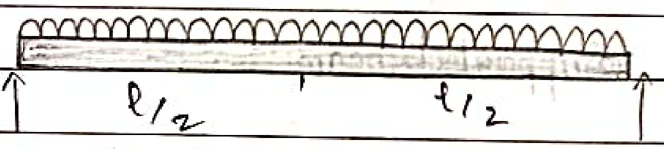
Statically Indeterminate

* Types of Loading :-

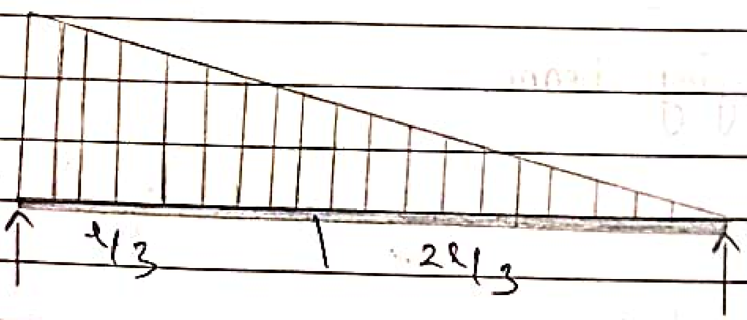
1. Concentrated or point load



2. Uniformly distributed load



3. Uniformly Varying load



* Shear force :- The Shear force at the cross-section of a beam, is defined as the unbalanced vertical force to the right or left of the section. Unit Newton or kilo Newton

Normal Stress
Direct Stress — CG
Bending Stress — Due to Bending

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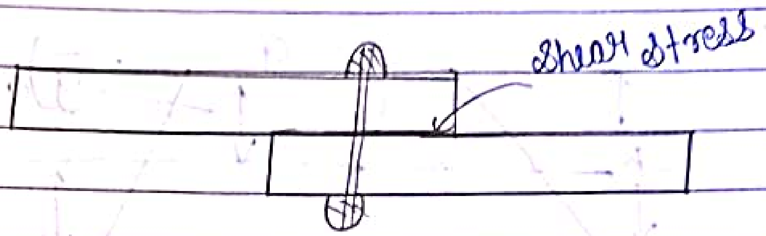
② Shear stress



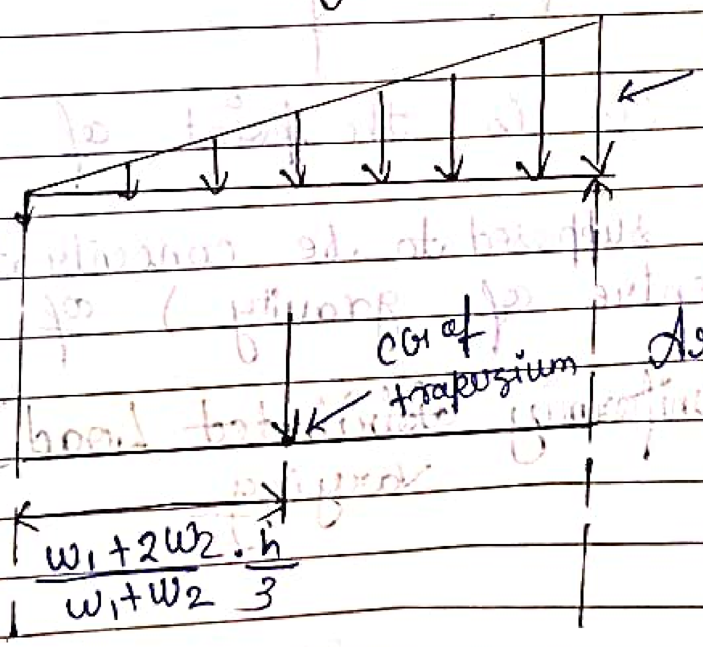
* **Bending Moment :-**

It is defined as the algebric sum of moments of forces acting on the left side or right side of the section.

Unit :- N-mm or 'kilo-mm'

$$B.M = Force \times distance$$

* **Trapezoidal loading.**



$$Area = \frac{1}{2}(w_1 + w_2) \cdot l$$

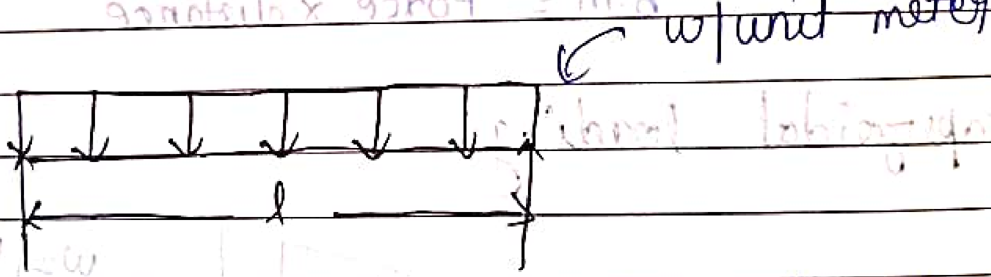
Area of Trapezium = $\frac{1}{2}$ (Sum of parallel sides) x altitude

Trick to find CG of Trapezium.

$$\bar{y})_a = \frac{1\cdot a + 2\cdot b}{a+b}\left(\frac{h}{3}\right)$$

$$\bar{y})_b = \frac{1\cdot b + 2\cdot a}{a+b}\left(\frac{h}{3}\right)$$

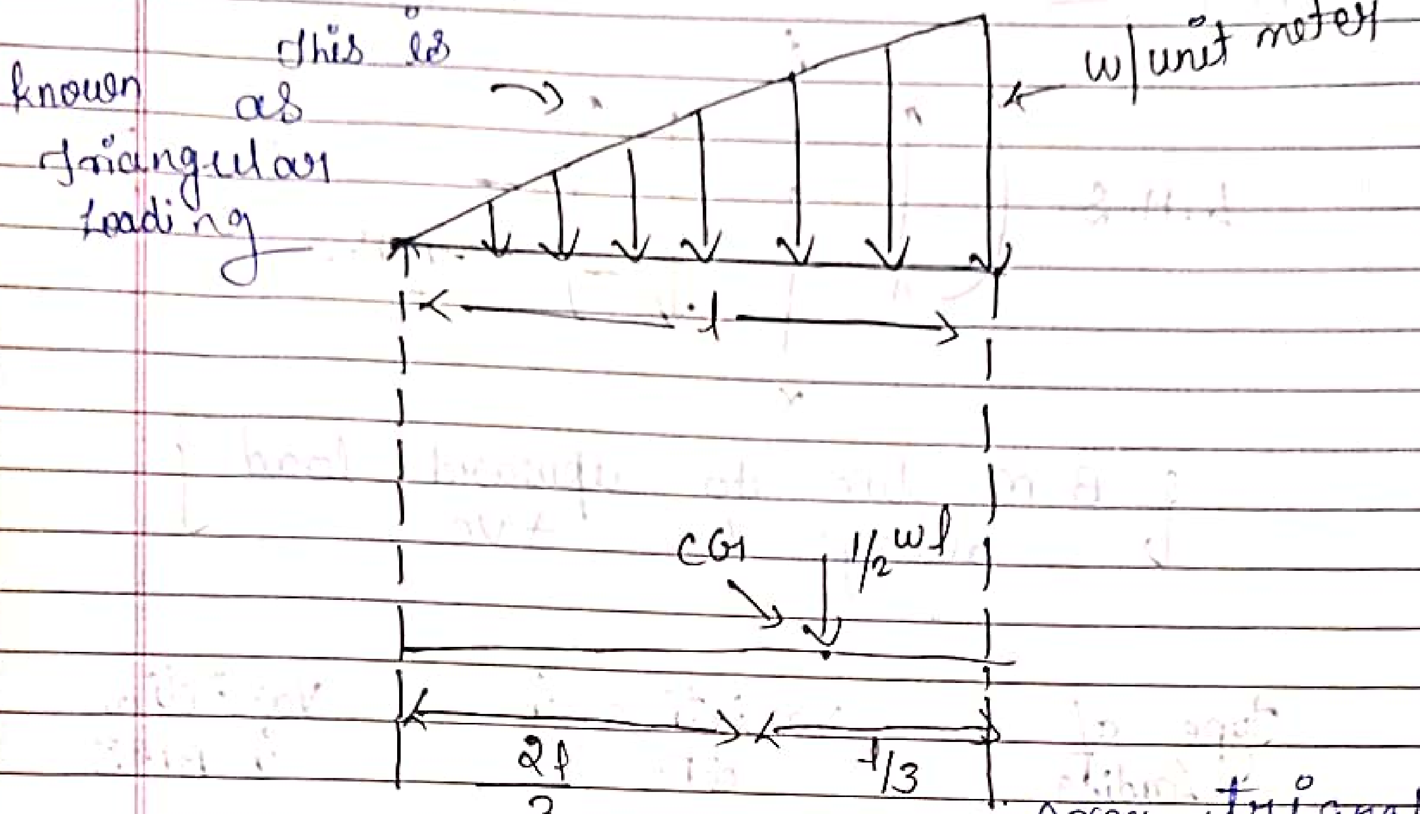
* UDL :-



In UDL, CG is the point of application.

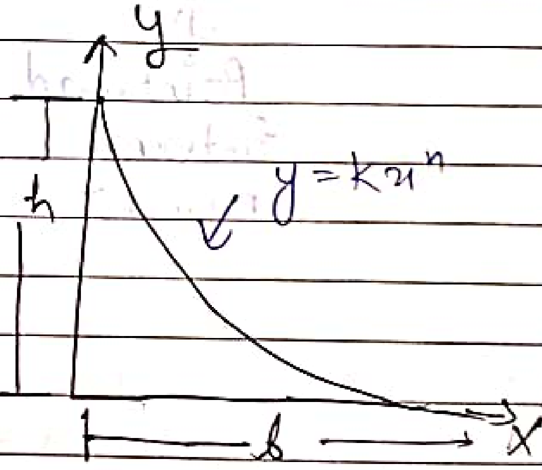
UDL is supposed to be concentrated at CG (centre of gravity) of the body

* UVL [uniformly ~~distributed~~ varying Load]



Area triangle

Total load $= \frac{1}{2} \times w\,l = \frac{wl}{2}$

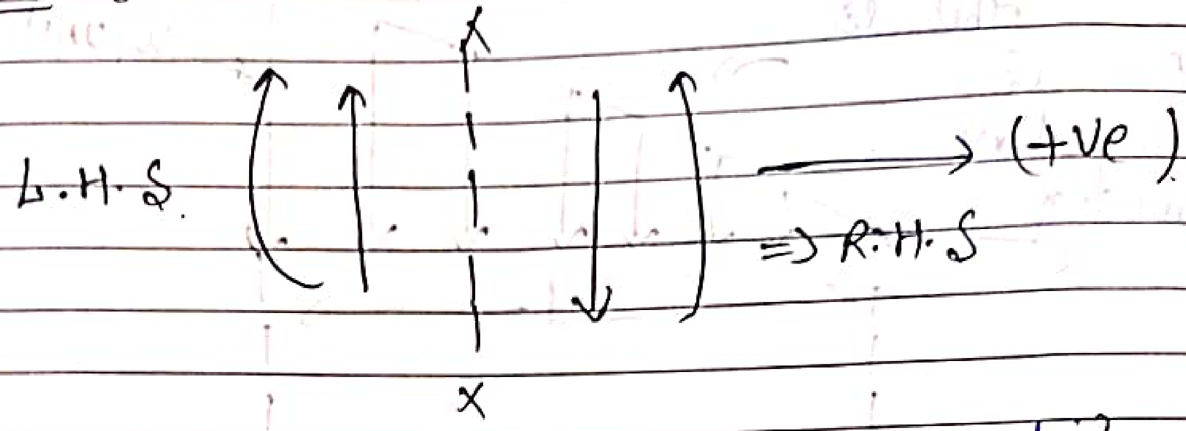


Area $= \frac{1}{n+1}\, b \cdot h$

CG from Right angle $\bar{x})_{R.L} = \frac{b}{n+2}$

$\bar{y})_{R.L} = \frac{h}{n+2}$

NOTE :—



{ B.M due to upward load always be +ve }

Type of Loading	Variation in SFD	Variation in BMD
No load	Horizontal line (0°C)	Inclined line (1°C
UDL	Inclined line (1°C)	Parabolic (2°C
UVL	Parabolic (2°C)	Cubic Parabolic
Point Load	Sudden change	No effect
External Moment	No effect	Sudden Change

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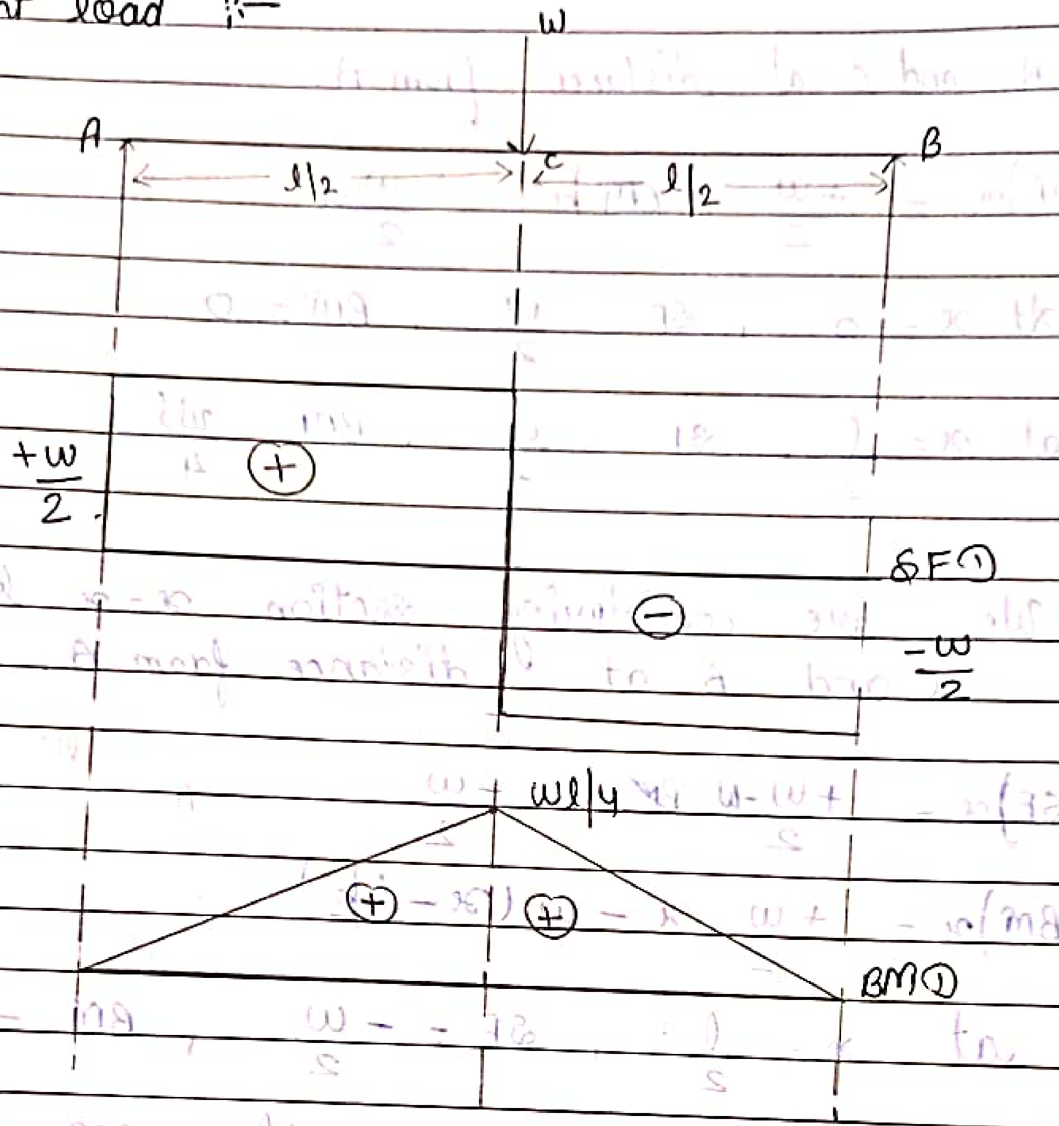
★. Point load :-



$\Sigma V = 0$

$R_A + R_B - W = 0$

$R_A + R_B = W \quad \text{——— (i)}$

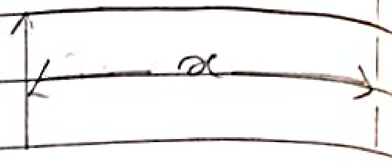
$(\Sigma M)_A = 0 \Rightarrow +W \times \frac{l}{2} - R_B \times l = 0$

$R_B = +\frac{W}{2}$

$R_A = +\frac{W}{2}$

We are considering section $x - x$ between

A and C at distance from A.

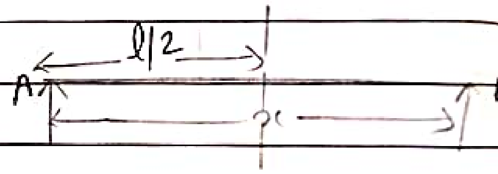


$SF)_x = \frac{+W}{2}$, $BM)_x = \frac{+W}{2}x$

At $x = 0$, $SF = \frac{W}{2}$, $BM = 0$

at $x = \frac{l}{2}$, $SF = \frac{W}{2}$, $BM = \frac{Wl}{4}$

We are considering section x - x between C and B at distance from A.



$SF)_x = \frac{+W}{2} - W$, $= \frac{-W}{2}$

$BM)_x = \frac{+W}{2} x - W(x - l/2)$

at $x = \frac{l}{2}$, $SF = -\frac{W}{2}$, $BM = \frac{Wl}{4}$

at $x = l$, $SF = \frac{-Wl}{2}$, $BM = 0$

Que :-

$R_A + R_B = 40 + 80$

$R_A + R_B = 120$

$(\Sigma M)_A = 0 = +40 \times 2 + 80 \times 2 - R_B \times 10$

$R_B = 56 \text{ kN}$

$R_A = 64 \text{ kN}.$

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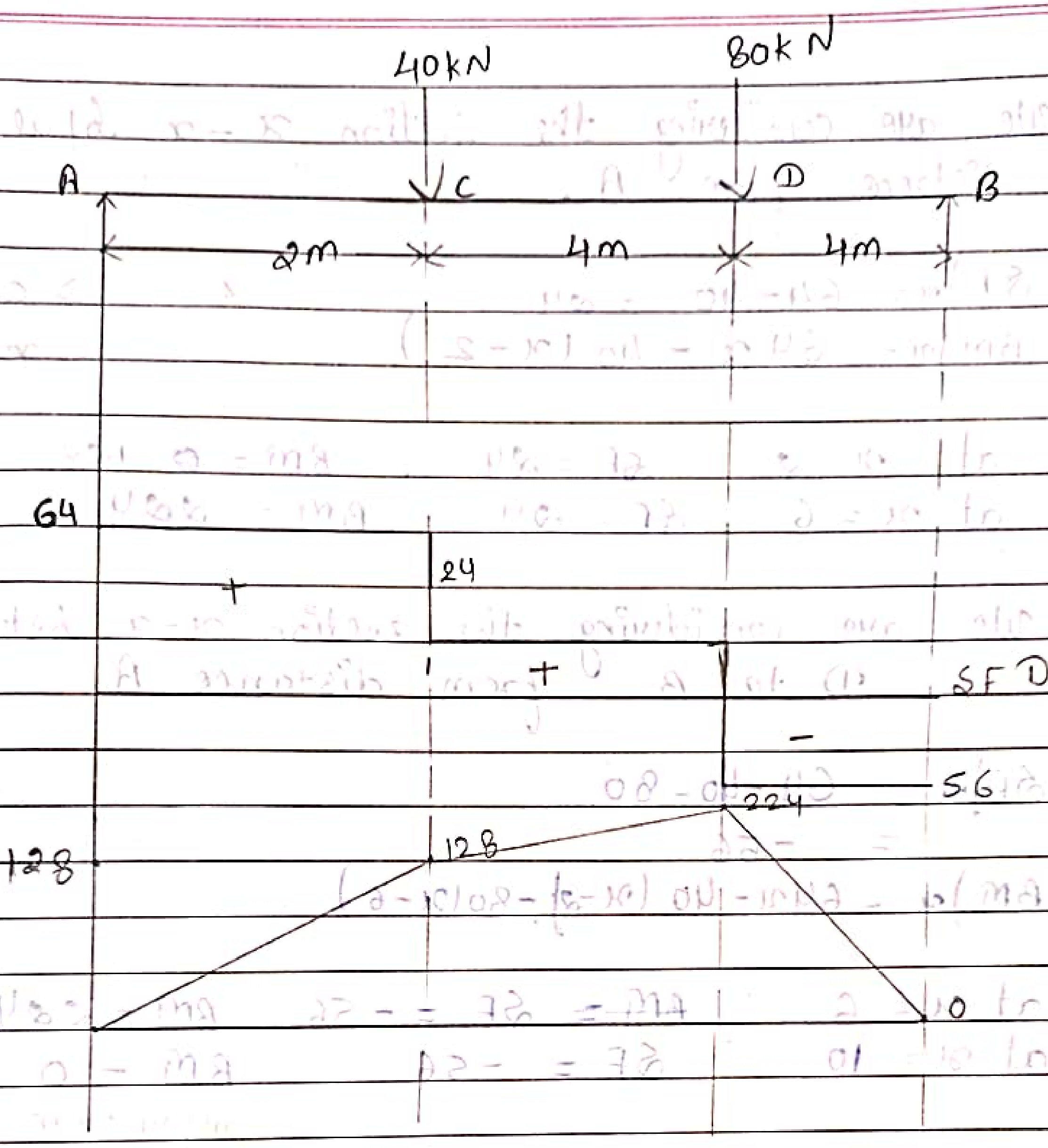
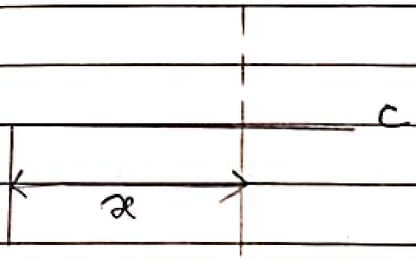


We are considering section x-x between A to C at distance from A.

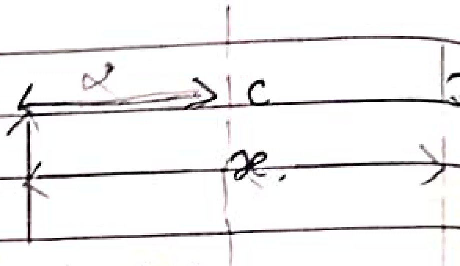
$SF)_x = 64$, $BM)_x = 64x$



at $x=0$, $SF = 64$, $BM = 0$

at $x=2$, $SF = 64$, $BM = 128$

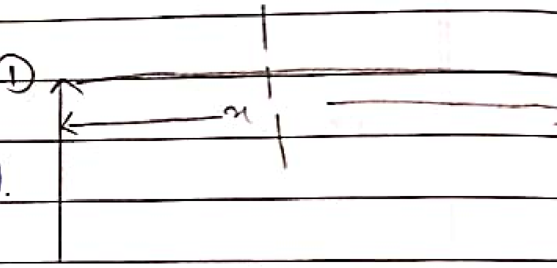
We are considering the section x – x b/w C to D distance from A.



$SF)_x = 64 - 40 = 24$
$BM)_x = 64x - 40(x - 2)$.

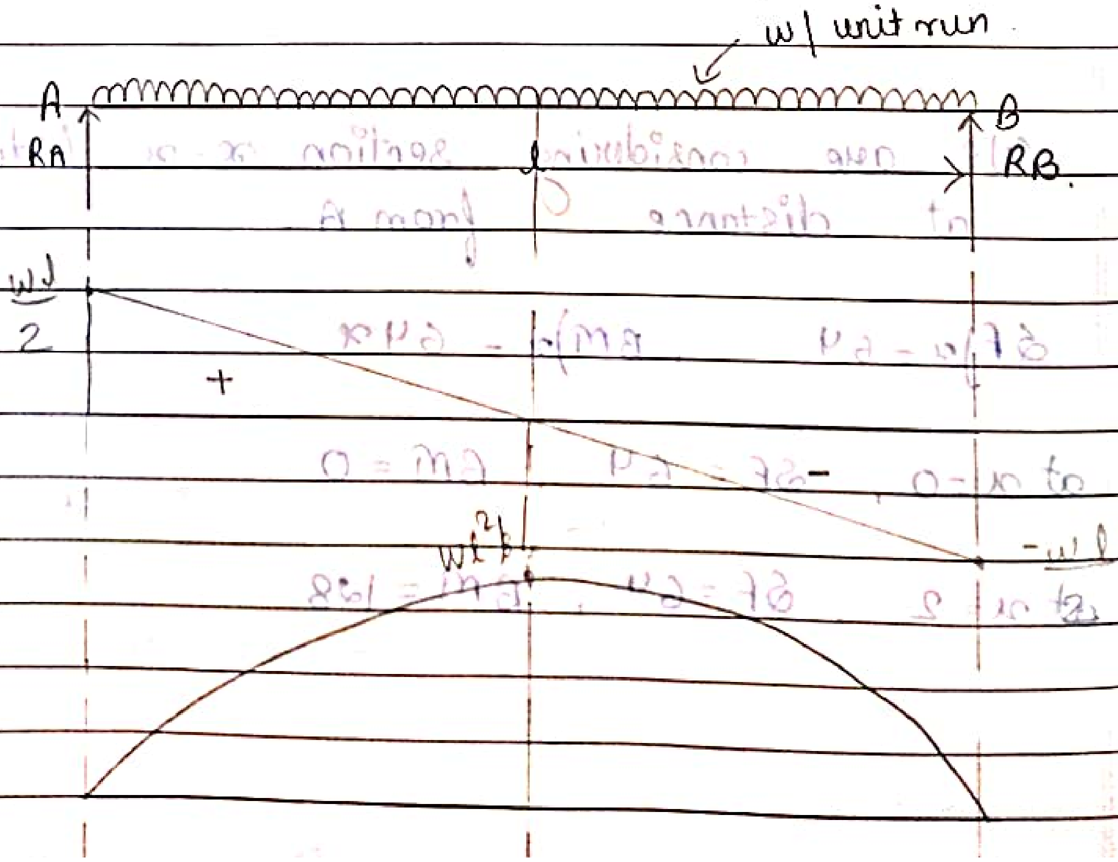
at $x = 2$ $SF = 24$, $BM = 128$
at $x = 6$ $SF = 24$, $BM = 224$.

We are considering the section x – x between D to B from distance A.



$SF)_x = 64 - 40 - 80$
$= -56$
$BM)_x = 64x - 40(x-2) - 80(x-6)$.

at $x = 6$ ~~BM =~~ $SF = -56$, $BM = 224$.
at $x = 10$ $SF = -56$ $BM = 0$



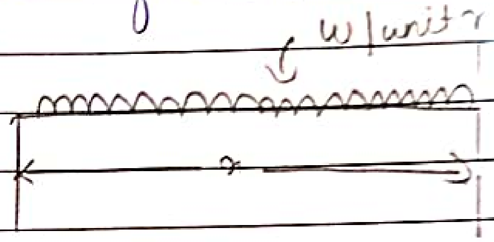
$\uparrow \Sigma V = 0$

$R_A + R_B = wl$ —— (i)

$(\circlearrowright \Sigma M)_A = 0 = wl\left(\frac{l}{2}\right) - R_B \times l$

$R_B = \frac{wl}{2}$, $R_A = +\frac{wl}{2}$

We are considering x - x section b/w A and B at distance from A.



$SF)_x = \frac{wl}{2} - wx$

$BM)_x = \frac{wl}{2}x - wx\left(\frac{x}{2}\right)$.

at $x = 0 \Rightarrow \frac{wl}{2} - 0 \Rightarrow \frac{wl}{2} = SF$, $BM = 0$

at $x = l \Rightarrow -\frac{wl}{2} = SF$, $BM = 0$

$\frac{wl}{2} - wx = 0 \Rightarrow x = \frac{l}{2}$

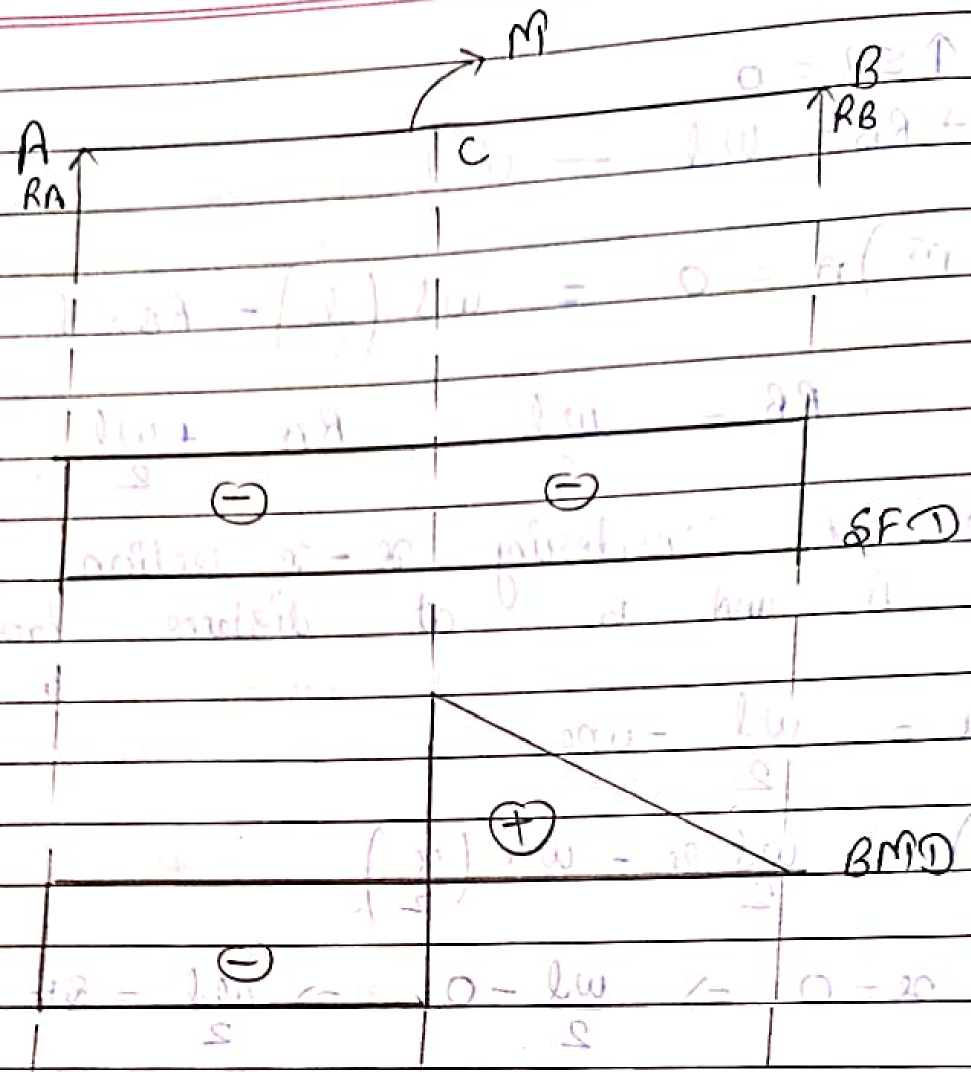
at $x = \frac{l}{2}$, $BM = B_{max}$

$BM)_x = \frac{wl}{2}\left(\frac{l}{2}\right) - \frac{w}{2}\left(\frac{l}{2}\right)^2$

$BM)_x = \frac{wl^2}{4} - \frac{wl^2}{8}$

$BM_{max})_x = \frac{wl^2}{8}$

Que :-



Support Reaches = RA = RB = $\frac{\text{Res. } M}{\text{distance bet}^n \text{ support}}$

$= RA = RB = \frac{M}{l}$

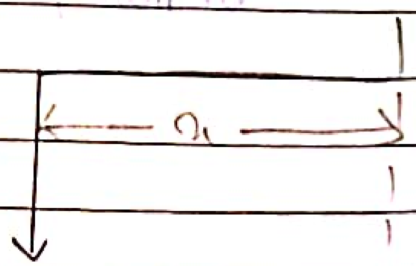
$RA + RB = 0$

$(\overset{\rightarrow}{\Sigma M})_A = 0 \Rightarrow M - R_B \times l = 0$

$RA = -\frac{M}{l}$

we are considering a section $x-x$ b/w A and C from A.

$SF)_x = \frac{-M}{l}$, $BM)_x = -\frac{M}{l} \cdot x \cdot \frac{M}{l}$

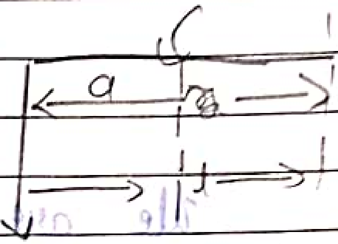


at $x=0$ $\quad S.F = \dfrac{-M}{l}$, $B.M = 0$

at $x=a$ $\quad S.F = \dfrac{-M}{l}$ $\quad BM = \dfrac{-Ma}{l}$.

We are considering a section $x - x$ b/w C and B distance from A.

$SF)_x = \dfrac{-M}{l}$, $BM)_x = -\dfrac{M}{l}x + M$



at $x = 0$, $SF = \dfrac{-M}{l}$, $BM = \dfrac{M}{l}x + M$.

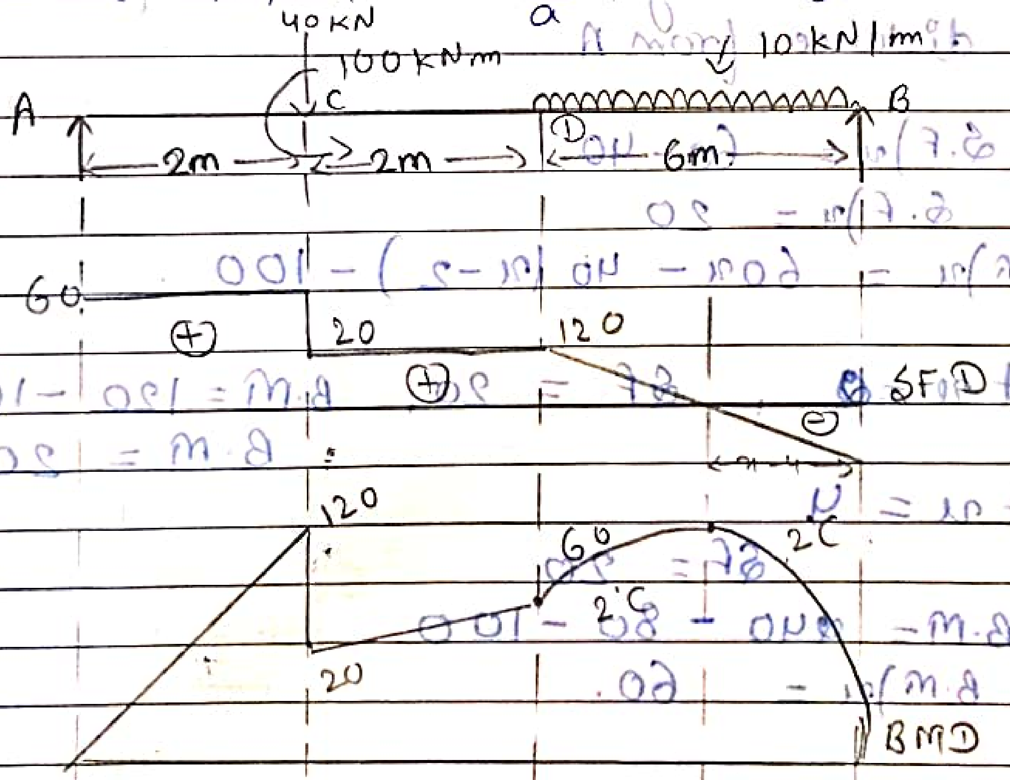
$= M\left(1 - \dfrac{a}{l}\right)$

$= M\left(\dfrac{l-a}{l}\right)$

$= M\left(\dfrac{b}{l}\right)$

at $x = l$, $SF = -\dfrac{M}{a}$, $BM = 0$.

Que.)



$\uparrow \Sigma V = 0$

$R_A + R_B = 40 + 10 \times 6$

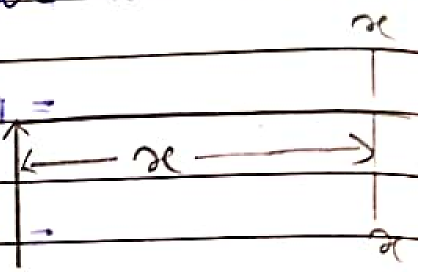
$R_A + R_B = 100$ —— (i)

$(\circlearrowright \Sigma M)_A = 0 \Rightarrow 40 \times 2 - 100 + (\underbrace{10 \times 6}_{T\text{-}UDL}) \times 7 - R_B \times 1$

$R_B = 40$

$R_A = 60$

We are considering section x – x between A and C, distance from A.



$S.F)_x = 60$

$B.M)_x = 60x$

at $x = 0$, $S.F = 60$, $B.M = 0$

at $x = 2$, $S.F = 60$, $B.M = 120$

We are considering section x – x b/w C and D distance from A.

$S.F)_x = 60 - 40$

$S.F)_x = 20$

$B.M)_x = 60x - 40(x-2) - 100$

at $x = 2$, $S.F = 20$, $B.M = 120 - 100$

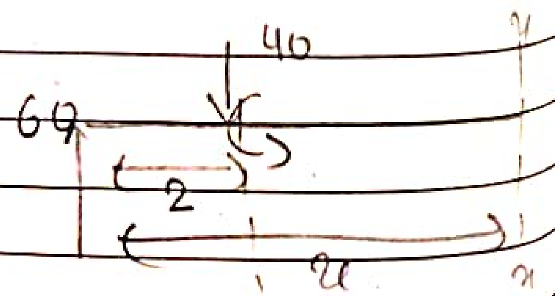
$B.M = 20$

at $x = 4$

$SF = 20$

$B.M = 240 - 80 - 100$

$B.M)_x = 60$



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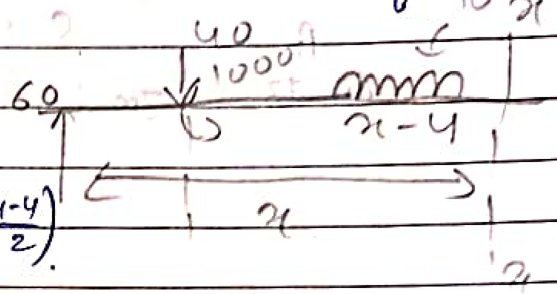
We are consider section $x \rightarrow x$ b/w D & B from A.



$S.F)_x = 60 - 40 - 10(x-4)$

$BM)_x = 60x - 40 \times 2 - 100 - 10(x-4)\left(\frac{x-4}{2}\right)$.

at $x = 4$, $SF = 20$, $BM = 240 - 80 - 100 - 0$
$BM = 60$.

~~We are considering section $x \rightarrow x$ b/w D and~~
at $x = 10$, $S.F = -40$.
$B.M = $ ~~600 - 800 - 100 - 10 (6). (3)~~

$B.M = 0$

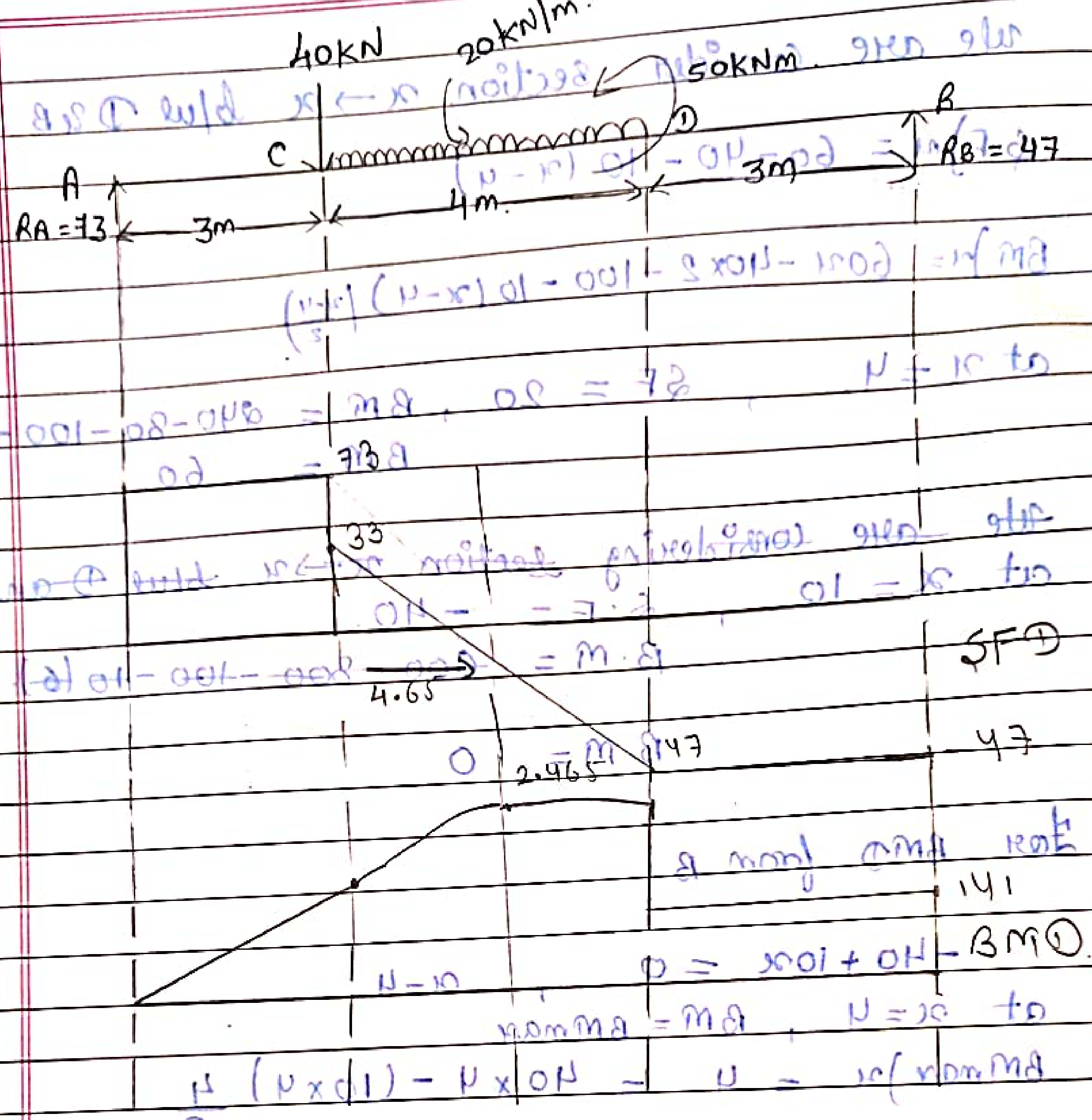
For BMD from B.

$-40 + 10x = 0$, $x = 4$.
at $x = 4$, $BM = BM_{max}$
$BM_{max})_{x} = 4 = 40 \times 4 - (10 \times 4)\frac{4}{2}$

$= 160 - 40 \times 2$

$= (160 - 80$

$= 080$



$\Sigma V = 0$

$R_A + R_B = 40 + (20 \times 4) =$

$R_A + R_B = 40 + 80 =$

$R_A + R_B = 120$. — (i)

$\Sigma M)_A = 40 \times 3 + (20 \times 4) \times 5 - 50 - R_B \times 10$

$\Sigma M)_A = 120 + 80 \times 5 - 50 = R_B \times 10$

$= 120 + 400 - 50$

$R_B = \frac{470}{10} = 47$

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$R_B = 47$

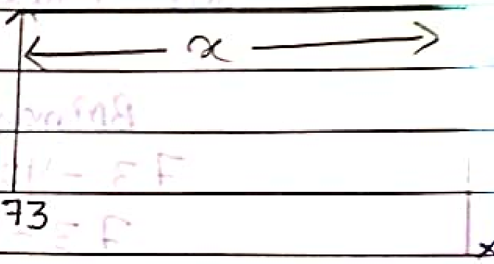
Putting these value in Eqn (i)

$R_B = 120 - 47$

$R_B = 73$

Consider the Section x−x between A to C from A.

$SF)_x = 73$

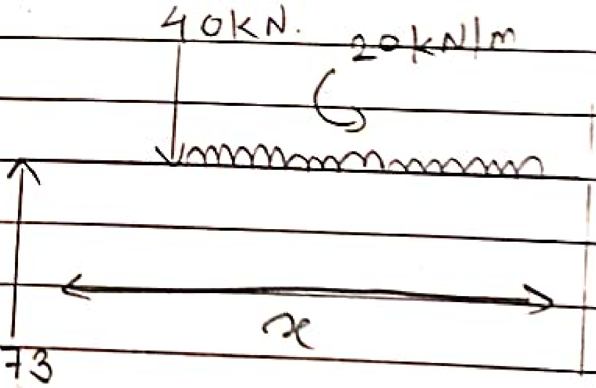


$BM)_x = 73x$

At $x=0$, $SF = 73$, $BM = 0$

At $x=3$, $SF = 73$, $BM = 219$.

We are considering the section x−x b/w C to D from A.



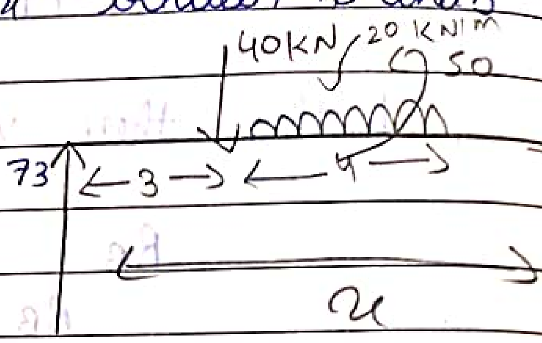
$SF)_x = 73 - 40 - 20(x-3)$.

$BM)_x = 73x - 40(x-3) - 20(x-3) \times \frac{x-3}{2}$

at $x = 3$, ~~SF = 219~~ $SF = 33$, $BM = 219$

at $x = 7$, $SF = -47$, $BM = 191$

Considering the section x–x between D and B from A



$SF)_x = 73 - 40 - 20(4)$

$SF)_x = -47$

$BM)_x = 73x - 40(x-3) - (20 \times 4)(x-5) - 50$

at $x = 7$, $SF = -47$, $BM = 141$

For Maximum B.M.

$BM_{max} =$

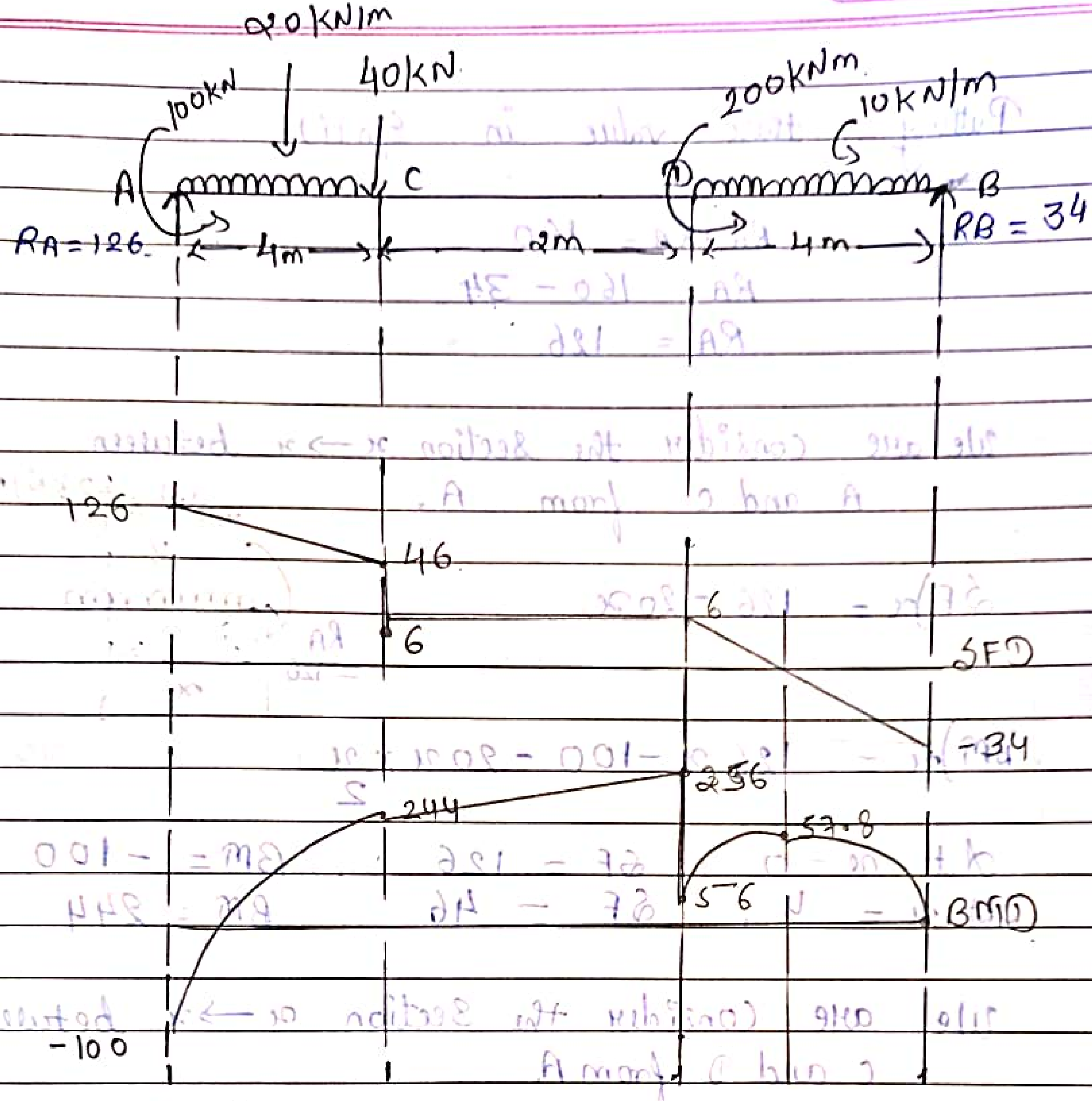
$73 - 40 - 20(x-3)$

$73 - 40 = 20(x-3)$

$\frac{33}{20} = x - 3$

$x = 4.65$

Que :-



$\Sigma V = 0$

$R_A + R_B = 20 + 40 - 20 \times 4 + 40 + 10 \times 4$

$R_A + R_B = 80 + 40 + 40$

$R_A + R_B = 160$ ——— (i)

$\Sigma M)_A = 0 \Rightarrow -100 + (20 \times 4) \times 2 + 40 \times 4 - 200 + (10 \times 4) \times 8 - R_B \times 10$

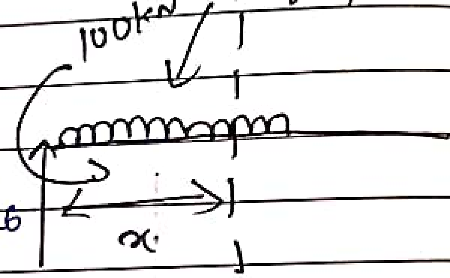
$R_B \times 10 = 340$

$R_B = \frac{340}{10} = 34.$

Putting these value in Eqn (i).

$$R_A + R_B = 160$$
$$R_A = 160 - 34$$
$$R_A = 126$$

We are consider the section $x \rightarrow x$ between A and C from A.

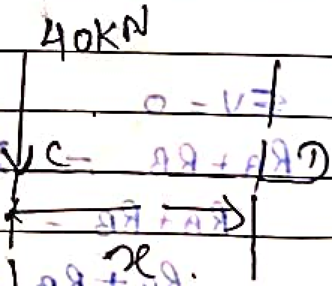


$SF)_x = 126 - 20x$

$BM)_x = 126x - 100 - 20x \times \frac{x}{2}$

At $x = 0$, $SF = 126$, $BM = -100$
At $x = 4$, $SF = 46$, $BM = 244$

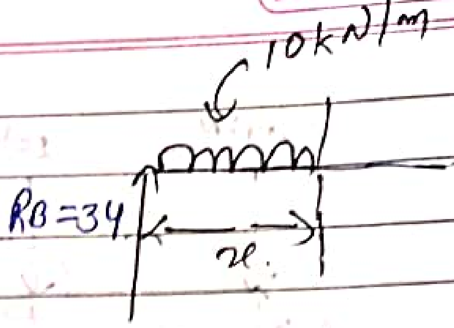
We are consider the section $x \rightarrow x$ between C and D from A.



$SF)_x = 126 - 20 \times 4 - 40 = 6$
$BM)_x = -100 - (20 \times 4)(x - 2) - 40(x - 4) + 126x$
~~BM)x =~~

At $x = 4$, $SF)_x = 6$, $BM = 244$
At $x = 6$, $SF)_x = 6$, $BM = 256$

We are consider the section $x \rightarrow x$ b/w B and D from B.



$SF)_x = -34 + 10x$

$BM_x = 34x - 10x \times \frac{x}{2}$

at $x = 4$, $SF = 6$, $BM = 56$

at $x = 0$, $SF = -34$, $BM = 0$.

For mid point

$-34 + 10x = 0$

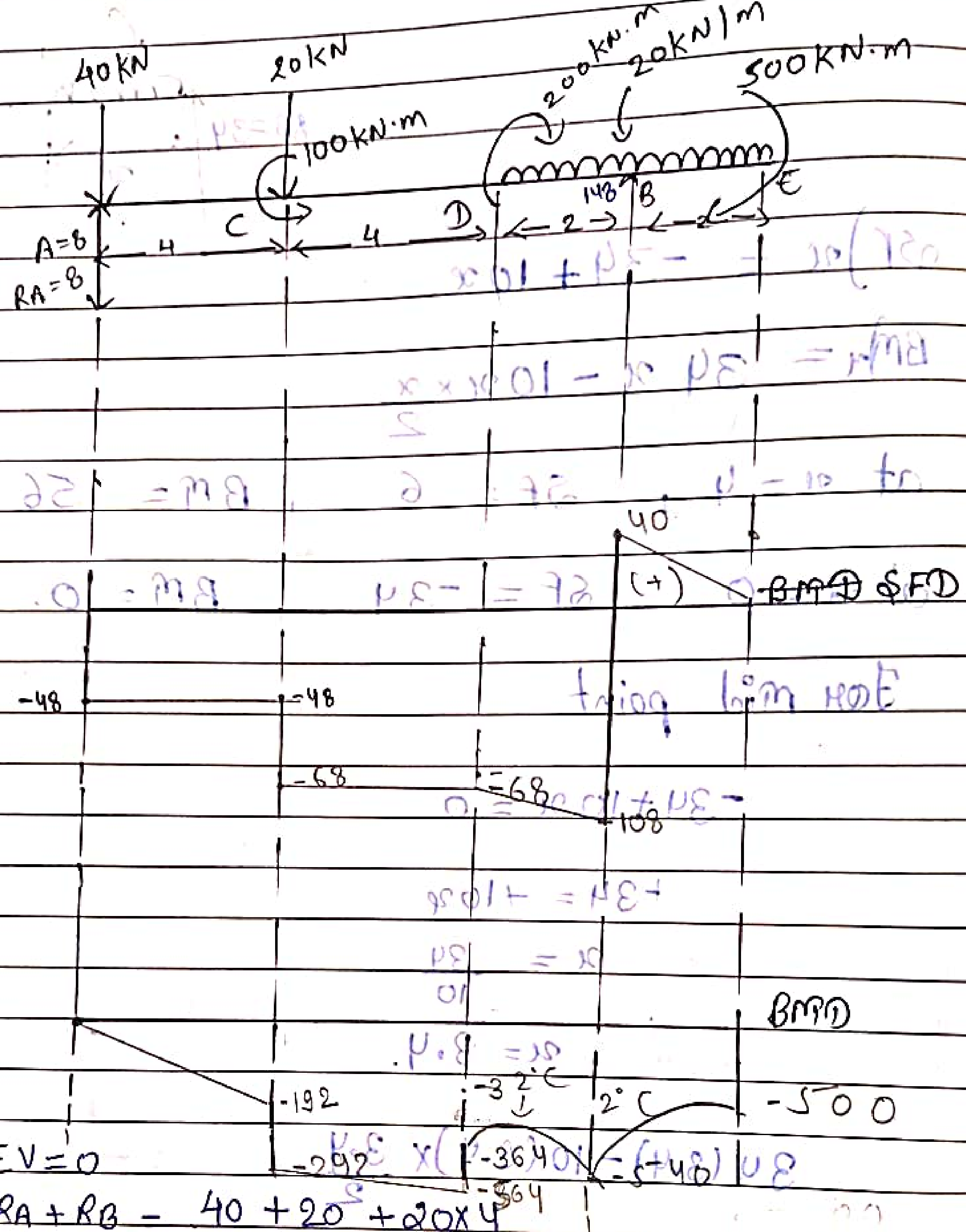
$+34 = +10x$

$x = \frac{34}{10}$

$x = 3.4$.

$34(3.4) - 10(3.4) \times \frac{3.4}{2}$

$= 57.8$



$\Sigma V = 0$

$R_A + R_B = 40 + 20 + 20 \times 4$

$R_A + R_B = 140$ —— (i)

$\Sigma M)_A = 0 \Rightarrow 20 \times 4 - 100 + 200 + 20 \times 4 \times 10 + 500 - R_B \times 10$

$R_B \times 10 = 1480$

$R_B = 148.$

In Eqn (i)

$R_B = 140 - 148 = -8$ kN.

↓ → SF = +ve
Anticlosewise = +ve } Right

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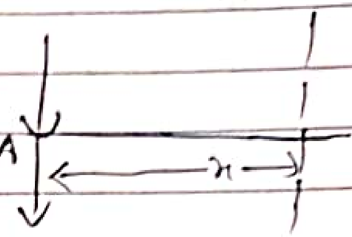
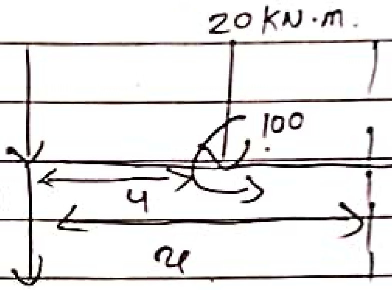
Consider the section x - x between A to C from A



$SF)_x = -8 - 40$
$SF)_x = -48$
$BM)_x = -48x$

At $x = 0$, S.F = -48, BM = 0
At $x = 4$, S.F = -48, BM = -192

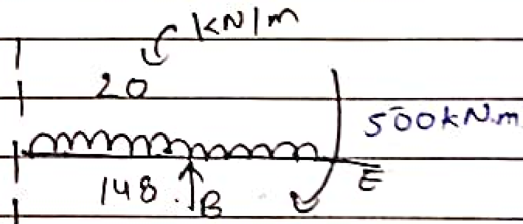
We are consider the section x-x b/w C & D from A



$SF)_x = -48 - 20 = -68$
~~SF~~ BM $)_x = -48x - 20(x-4) - 100$

at $x = 4$, SF = -68, BM = -292
at $x = 8$, SF = -68, BM = -564

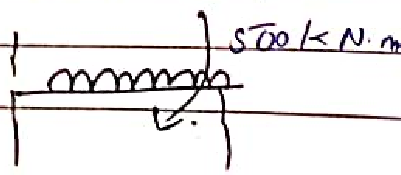
We are consider the section x-x b/w D & B from E



$SF)_x = -148 + 20x$
~~$BM)_x = -148x - 20x \times 3 - 500$~~
$BM)_x = +148(x-2) - 20x \times \frac{x}{2} - 500$

at $x = $ ~~6~~, $SF)_x = $ ~~12~~ -68, BM = -364
at $x = 2$, $SF)_x = -108$, BM = -540.

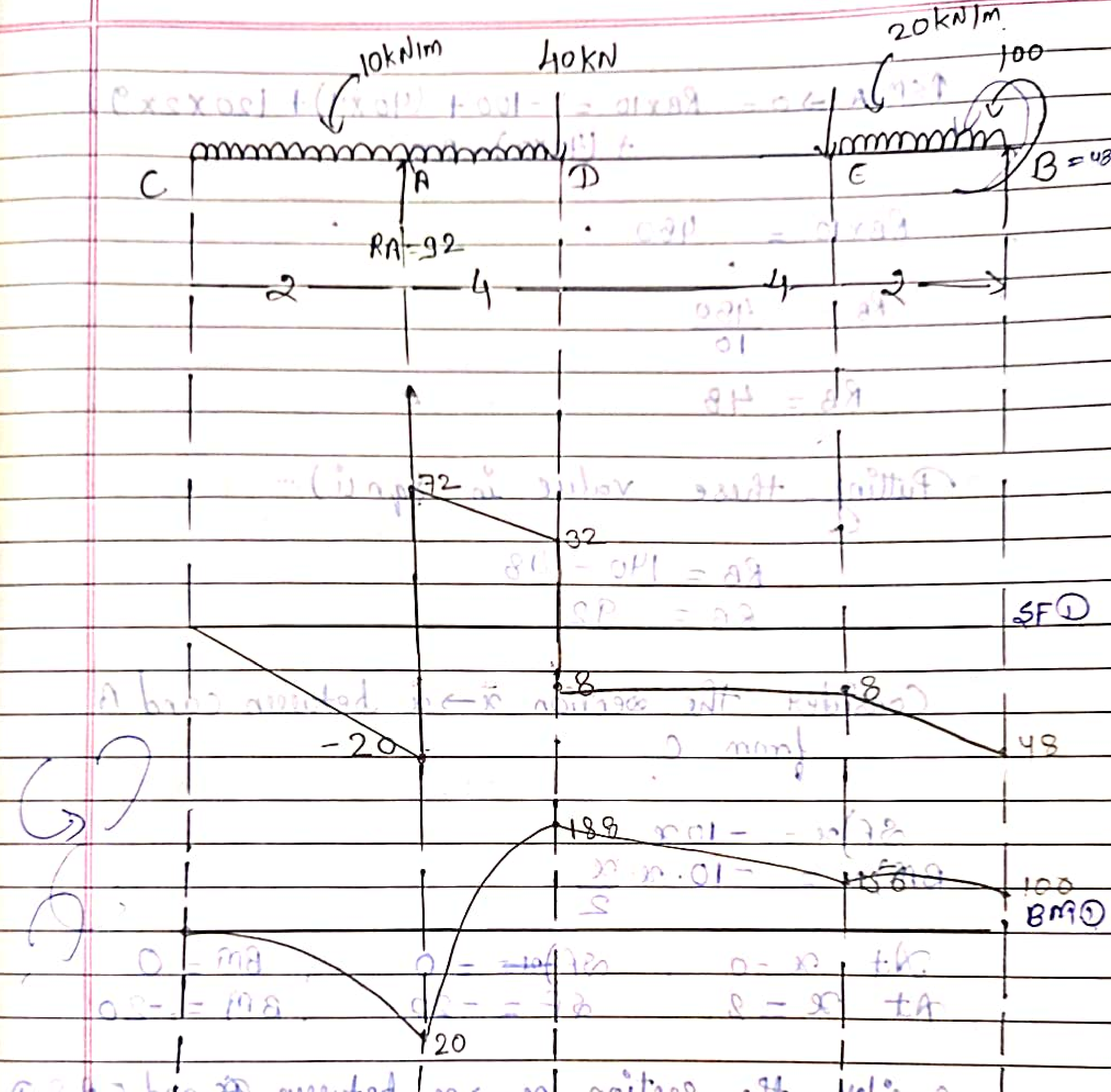
We are consider the section x-x b/w B & E from E



$SF)_x = 20x$
$BM = -500$ ~~+20~~ $- 20x \times \frac{x}{2}$

at $x = 2$, $SF = 40$, $BM = -540$.

at $x = 0$, $SF = 0$, $BM = -500$.



$\Sigma V = 0$

$R_A + R_B = 10 \times 6 + 40 + 20 \times 2$

$R_A + R_B = 60 + 40 + 40$

$R_A + R_B = 140$ ——— (i)

Taking Moment about A

$\uparrow \Sigma M)_A \Rightarrow 0 = R_B \times 10 = -100 + (40 \times 4) + (20 \times 2 \times 9) + (40 \times 2) - 20$

$R_B \times 10 = 480$

$R_B = \frac{480}{10}$

$R_B = 48$

Putting these value in Eqn (i)

$R_B = 140 - 48$

$R_B = 92$

Consider the section $x \to x$ between c and A from C.

$SF)_x = -10x$

$BM)_x = -10 \cdot x \cdot \frac{x}{2}$

At $x = 0$, $SF = 0$, $BM = 0$

At $x = 2$, $SF = -20$, $BM = -20$

Consider the section $x \to x$ between ~~B and~~ E A & D from C

$SF)_x = 92 - 10x$

$BM)_x = 92(x-2) - 10 \cdot x \cdot x$

At $x = 2$, $SF = 72$, $B.M. = -20$

At $x = 6$, $SF = 32$, $BM = 188$

Consider the section $x \to x$ b/w D & E from C

$SF)_x = 92 - 10x - 40$

$BM)_x = 92(x-2) - 10\times 6(x-3) - 40(x-6)$

at $x = 6$, $SF = -8$, $BM = 188$

at $x = 10$, $SF = -48$, $BM = 156$

Consider the section $x \to x$ b/w B & E from B

$SF)_x = -48 + 20x$

$BM)_x = +100 + 48x - 20x \times \frac{x}{2}$

at $x = 2$, $SF = -8$, $BM = 156$

at $x = 0$, $SF = -48$, $BM = 100$.

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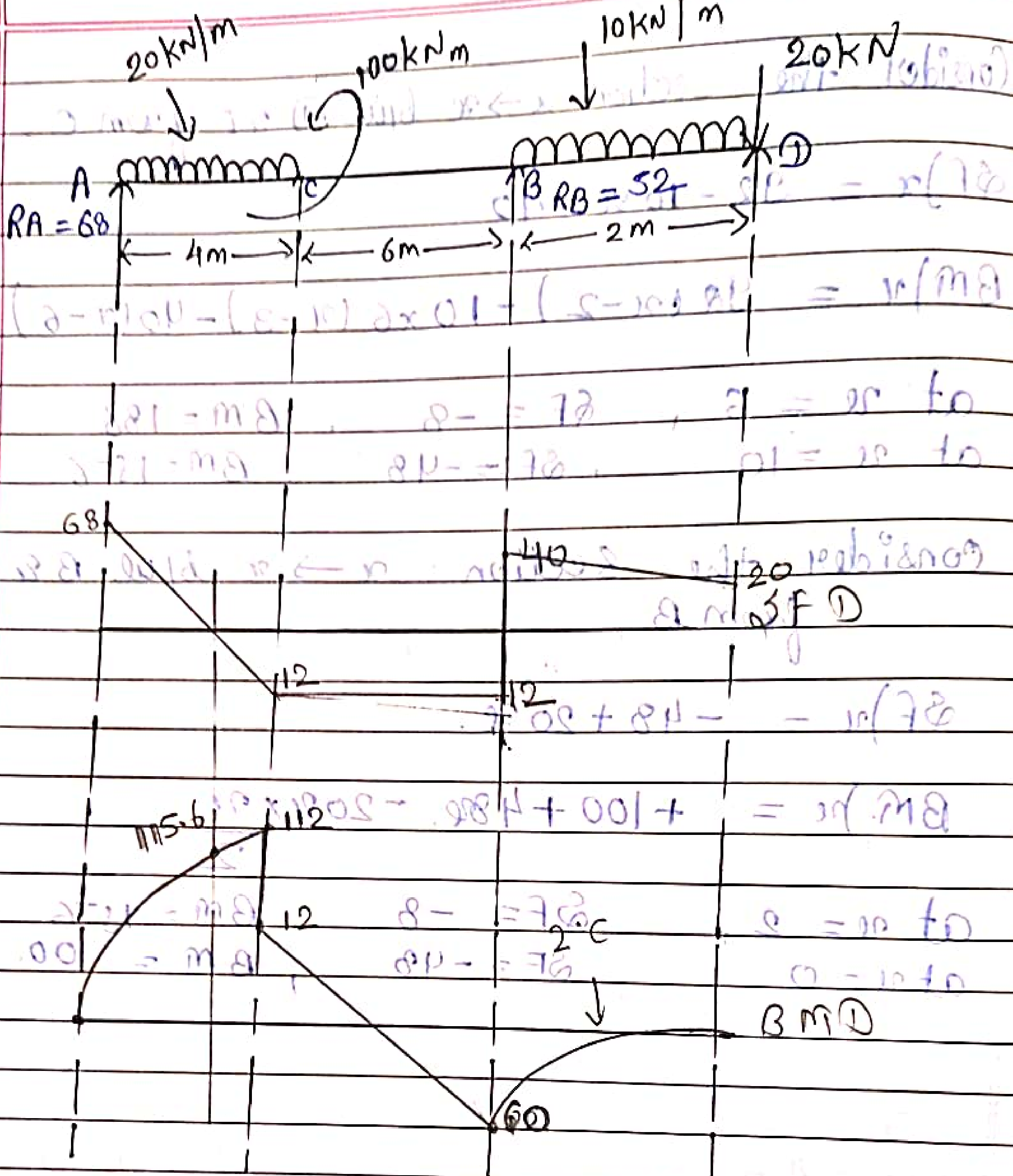


$\uparrow \Sigma V = 0$

$R_A + R_B = 20 \times 4 + 10 \times 2 + 20$

$= 80 + 20 + 20$

$R_A + R_B = 120$ —— (i)

$\uparrow \Sigma M)_A \Rightarrow 0 \Rightarrow R_B \times 10 = 20 \times 4 \times 2 - 100 + 10 \times 2 \times 11 + 20 \times 12$

$R_B \times 10 = 520$

$R_B = \frac{~~240~~}{~~10~~} \quad \frac{520}{10}$

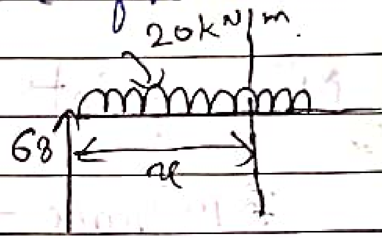
$R_B =$ ~~24~~ 52.

Putting these value in Eqn (i)

$R_A =$ ~~120 − 24~~ $120 - 52$

$R_A = 68$

Consider the section A and C from A b/w $x \rightarrow x$.

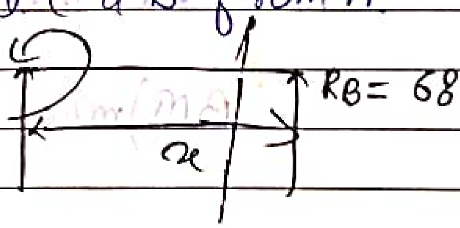


$SF)_x = 68 - 20x$

$BM)_x = 68x - 20x \cdot \frac{x}{2}$

at $x = 0$, $SF = 68$, $BM = 0$
at $x = 4$, $SF = -12$, $BM = 112$

Consider the section $x \rightarrow x$ b/w C & B from A



$SF)_x = 68 - 20 \times 4$
$= 68 - 80 = -12$

$BM)_x = 68x - 80(x-2) - 100$

At $x = 4$, $SF = -12$, $BM = 12$
At $x = 10$, $SF = -12$, $BM = -60$

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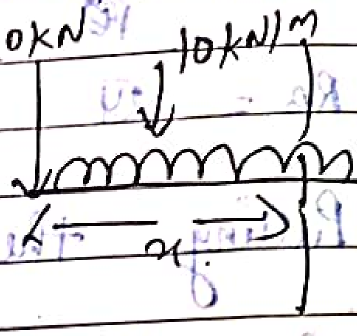
Consider the section x → x b/w B and D from D



$SF)_x = 20 + 10x$

$BM)_x = -20x - 10x \times \frac{x}{2}$

at $x = 2$, $SF = 40$, $BM = -60$
at $x = 0$, $SF = 20$, $BM = 0$.

Mid point

$BM)_{max} = 68 - 20x = 0$

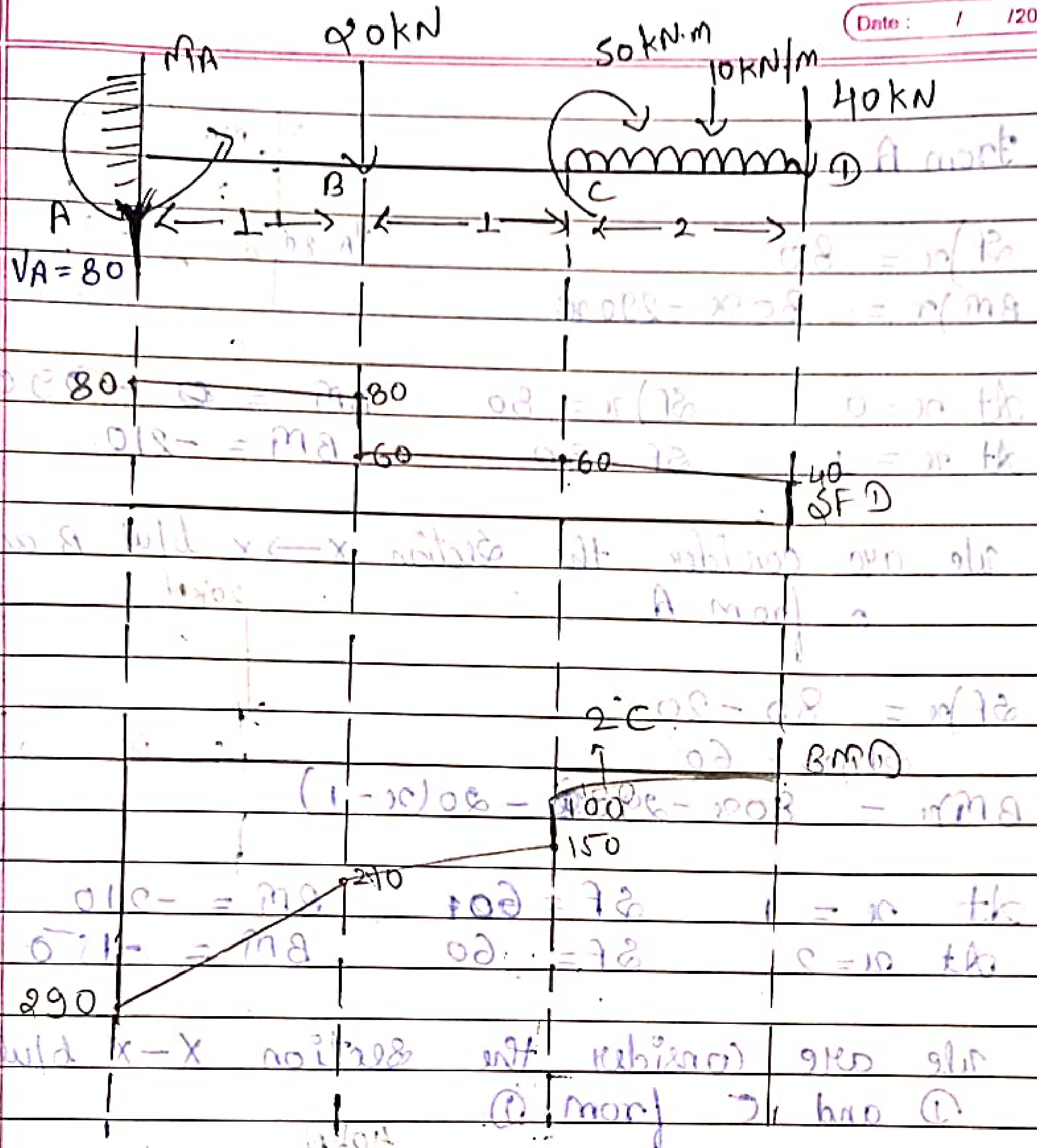
$68 = 20x$

$\frac{68}{20} = x$

$x = 3.4$

$BM)_{max} = 68 \times 3.4 - 20 \times 3.4 \times 1.7$

$BM)_{max} = 115.6$



$\Sigma V = 0 \Rightarrow V_A = 20 + 10 \times 2 + 40$

$= 20 + 20 + 40$

$= 40 + 40$

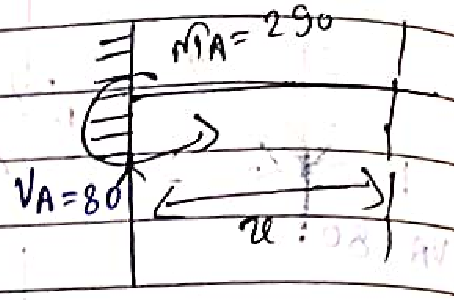
$= 80$

$\uparrow \Sigma M = -M_A + 20 + 50 + 40 \times 4 + (20 \times 3)$

$M_A = 290$

We are Consider the Section x→x between A and B

From A



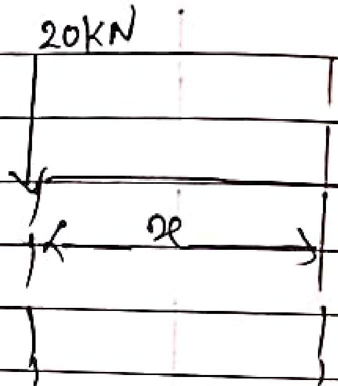
$SF)_x = 80$

$BM)_x = 80x - 290$

At $x = 0$ $SF)_x = 80$ $BM = -290$

At $x = 1$ $SF = 80$ $BM = -210.$

We are consider the section $x \to x$ b/w B and C from A.



$SF)_x = 80 - 20$

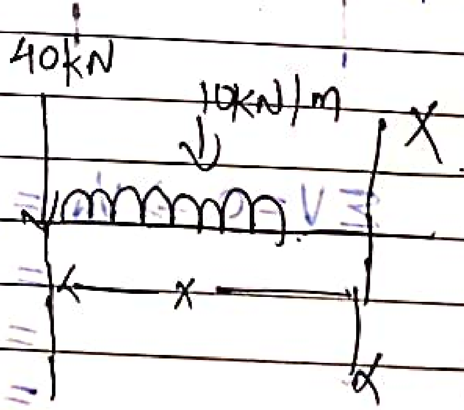
$SF)_x = 60$

$BM)_x = 80x - 290 - 20(x-1)$

At $x = 1$ $SF = 60$ $BM = -210$

At $x = 2$ $SF = 60$ $BM = -150$

We are consider the section $X - X$ b/w D and C from D.

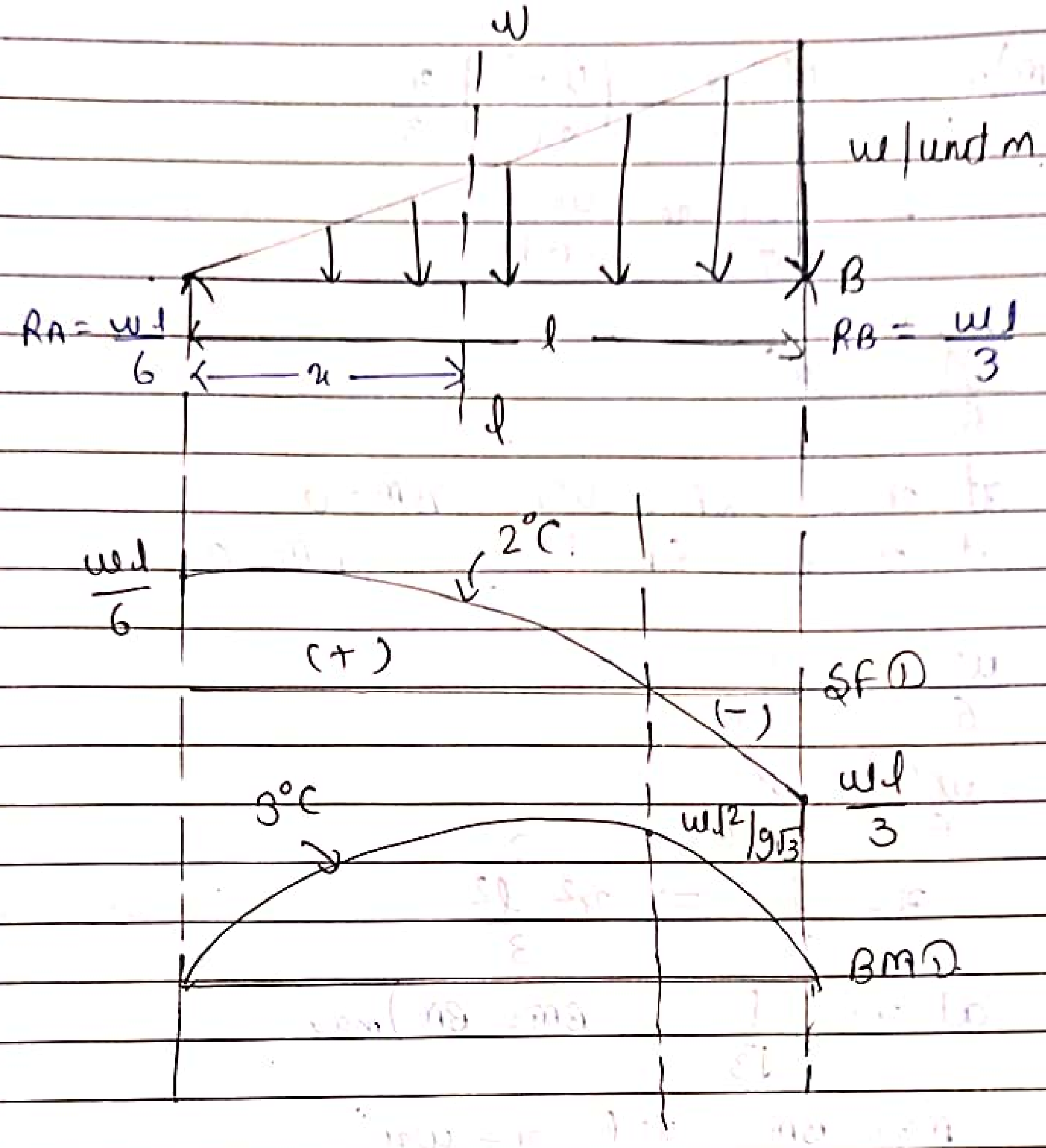


$SF)_x = 40 + 10x$

$BM = -40x - 10x \cdot \frac{x}{2}$

At $x = 2$ $SF = 60$ $BM = -100$

At $x = 0$ $SF = 40$ $BM = 0$



$$\uparrow \Sigma V = 0 \Rightarrow R_A + R_B = \frac{1}{2} w x \quad \text{— (i)}$$

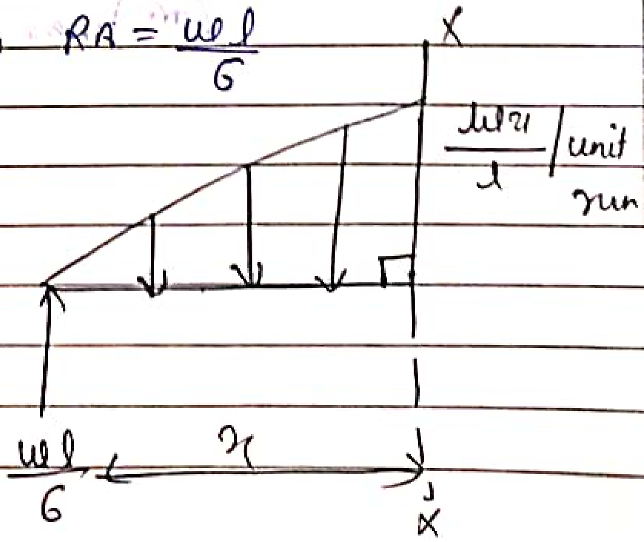
$$(\Sigma M)_A = 0 \Rightarrow \left(\frac{1}{2} w l\right)\left(\frac{2l}{3}\right) - R_B \times l = 0$$

$$R_B = \frac{wl}{3}, \quad R_A = \frac{wl}{6}$$

A & B, x from A

$$SF)_x = \frac{wl}{6} - \frac{1}{2}\frac{wx}{l}x$$

$$= \frac{wl}{6} - \frac{wx^2}{2l}.$$



$$BM)_x = \frac{wl}{6} \cdot x - \left(\frac{wx^2}{2l}\right)\frac{x}{3}$$

$$= \frac{wl}{6} x - \frac{wx^3}{6l}$$

$$\frac{wl}{6} - \frac{wx^2}{2l}$$

at $x=0$, $SF = \frac{wl}{6}$, $BM=0$

at $x=l$, $SF = \frac{-wl}{3}$, $BM=0$

$$\frac{wl}{6} - \frac{wx^2}{2l} = 0$$

$$\frac{wl}{6} = \frac{wx^2}{2l}$$

$$x = \frac{l}{\sqrt{3}} \Rightarrow x^2 = \frac{l^2}{3}$$

at $x = \frac{l}{\sqrt{3}}$, $BM = BM)_{max}$

$$\text{max } BM = \frac{wl}{6} x - \frac{wx^3}{6l}$$

$$= \frac{wl}{6}\left(\frac{l}{\sqrt{3}}\right) - \frac{w}{6l}\left(\frac{l}{\sqrt{3}}\right)^3$$

$$BM)_{max} = \frac{wl^2}{9\sqrt{3}}$$